Step 5. Repeat steps 3 and 4 till all allocations have been made. Successive reduced penalty matrices are obtained. Since the largest penalty (21) is now associated with the cell (1, 1), so allocate $x_{11} = 5$. This allocation ($x_{11} = 5$) eliminates the column 1 giving the second reduced matrix (*Table* 11·17).

Table 11·17

	W_3	W_4	Available	Penalty
F_1	*(50)	*(10)	2 (Note)	(40)
F_2	*(40)	*(60)	9	(20)
F_3	*(70)	10(20)	10/0	(50) ←
Requirements :	7	14/4		
Penalties :	(10)	(10)		

The largest penalty (50) is now associated with the cell (3, 4) therefore allocate $x_{34} = 10$. Eliminating the row 3, the third reduced penalty matrix ***Table* 11·18** is obtained.

Table 11·18

	W_3	W_4	Available	Penalties
F_1	*(50)	2(10)	2/0	(40)
F_2	7(40)	2(60)	9/0	(20)
Requirements :	7	4/0 (Note)		
Penalties :	(10)	(50) ↑		

Now, allocate according to the largest penalty (50) as $x_{14} = 2$ and remaining $x_{24} = 2$. Then allocate $x_{23} = 7$.

Step 6. Finally, construct ***Table* 11·19** for the required feasible solution.

The total cost is :
5(19) + 8(8) + 2(10) + 2(60) + 10(20) + 7(40) = Rs. 779.
This cost is Rs. 35 less as compared to the cost obtained by *Lowest Cost Entry Method.*

Table 11·19

	W_1	W_2	W_3	W_4	Available
F_1	5(19)			2(10)	7
F_2			7(40)	2(60)	9
F_3		8(8)		10(20)	18
Requirements :	5	8	7	14	

In order to reduce large number of steps required to obtain the optimal solution, it is advisable to proceed with the initial feasible solution which is close to the optimal solution. Vogel's method often gives the better initial feasible solution to start with. Although Vogel's method takes more time as compared to other two methods, but it reduces the time in reaching the optimal solution.

Short-cut. After a little practice, students may prefer to perform the entire procedure of Vogel's method within the original cost requirement *Table 11·14*. It needs merely to cross-out rows and columns as and when they are completed and to revise requirements, available supplies and penalties as shown below.

	W_1	W_2	W_3	W_4	Available	Penalties
F_1	5 • (19)	(30)	(50)	2 • (10)	7/2/0	(9/9/40/40)
F_2	(70)	(30)	7 • (40)	2 • (60)	9/0	(10/20/20/20)
F_3	(40)	8 • (8)	(70)	10 • (20)	18/10/0	(12/20/50)
Required	5/0	8/0	7/0	14/4/0		
Penalties	(21/21)	(22) ↑	(10/10/10/10)	(10/10/10/50)		

Example 2. *Obtain an initial basic feasible solution to the following transportation problem* :

		Stores				
		I	II	III	IV	Availability
	A	7	3	5	5	34
Warehouses	B	5	5	7	6	15
	C	8	6	6	5	12
	D	6	1	6	4	19
Demand		21	25	17	17	80

[Kanpur (B.Sc.) 2003; M.G. Univ., (M. Com.) 98]

[**Ans.** By Vogel's (Penalty) Method, the initial solution is :
$x_1 = 6$, $x_{12} = 6$, $x_{13} = 7$, $x_{14} = 5$, $x_{21} = 15$, $x_{34} = 12$, $x_{42} = 19$; Total transportation cost = Rs. 324.]

Q. Explain the use of Vogel's Approximation Method (VAM) with an example.

11·8-2. Summary of Methods for Initial BFS

The methods for obtaining an initial basic feasible solution to a transportation problem can be summarized as follows :

I. North-West Corner Rule (Stepping Stone Method)

Step 1. The first assignment is made in the cell occupying the upper left-hand (north-west) corner of the transportation table. The maximum possible amount is allocated there. That is, $x_{11} = \min(a_1, b_1)$. This value of x_{11} is then entered in the cell (1, 1) of the transportation table.

Step 2. *(i)* If $b_1 > a_1$, move vertically downwards to the second row and make the second allocation of amount $x_{21} = \min(a_2, b_1 - x_{11})$ in the cell (2, 1).

(ii) If $b_1 < a_1$, move horizontally right-side to the second column and make the second allocation of amount $x_{12} = \min(a_1 - x_{11}, b_2)$ in the cell (1, 2).

(iii) If $b_1 = a_1$, there is a tie for the second allocation. One can make the second allocation of magnitude $x_{12} = \min(a_1 - a_1, b_2) = 0$ in the cell (1, 2) or $x_{21} = \min(a_2, b_1 - b_1) = 0$ in the cell (2,1).

Step 3. Start from the new north-west corner of the transportation table and repeat steps 1 and 2 until all the requirements are satisfied.

II. The Row Minima Method

Step 1. The smallest cost in the first row of the transportation table is determined. Let it be c_{ij}. Allocate as much as possible amount $x_{1j} = \min(a_1, b_j)$ in the cell $(1, j)$, so that either the capacity of origin O_1 is exhausted, or the requirement at destination D_j is satisfied or both.

Step 2. *(i)* If $x_{1j} = a_1$ so that the availability at origin O_1 is completely exhausted, cross-out* the first row of the table and move down to the second row.

(ii) If $x_{1j} = b_j$ so that the requirement at destination D_j is satisfied, cross-out the jth column and re-consider the first row with the remaining availability of origin O_1.

(iii) If $x_{1j} = a_1 = b_j$, the origin capacity a_1 is completely exhausted as well as the requirement at destination D_j is completely satisfied. An arbitrary tie-breaking choice is made. Cross-out the jth column and make the second allocation $x_{1k} = 0$ in the cell $(1, k)$ with c_{1k} being the new *minimum* cost in the first row. Cross-out the first row and move down to the second row.

Step 3. Repeat *steps 1 and 2* for the reduced transportation table until all the requirements are satisfied.

III. The Column Minima Method

Step 1. Determine the smallest cost in the first column of the transportation table. Let it be c_{i1}. Allocate $x_{i1} = \min(a_i, b_1)$ in the cell $(i, 1)$.

* By saying *"crossout a row or a column"* we shall mean that no cell from that row or column can be chosen for the basis entry at a later step.

Step 2. *(i)* If $x_{i1} = b_1$, cross-out the first column of the transportation table and move towards right to the second column.

(ii) If $x_{i1} = a_i$, cross-out the ith row of the transportation table and reconsider the first column with the remaining demand.

(iii) If $x_{i1} = b_j = a_i$, cross-out the ith row and make the second allocation $x_{k1} = 0$ in the cell $(k, 1)$ with c_{k1} being the new minimum cost in the first column. Cross-out the column and move towards right to the second column.

Step 3. Repeat *steps 1 and 2* for the reduced transportation table until all the requirements are satisfied.

IV. Lowest Cost Entry Method (LCEM) or Matrix Minima Method

Step 1. Determine the smallest cost in the cost matrix of the transportation table. Let it be (c_{ij}). Allocate $x_{ij} = \min(a_i, b_j)$ in the cell (i, j).

Step 2. *(i)* If $x_{ij} = a_i$, cross-out the ith row of the transportation table and decrease b_j by a_i. Go to *step* 3.

(ii) If $x_{ij} = b_j$, cross out the jth column of the transportation table and decrease a_i by b_j. Go to *step* 3.

(iii) If $x_{ij} = a_i = b_j$, cross-out either the ith row or jth column but not both.

Step 3. Repeat *steps 1 and 2* for the resulting reduced transportation table until all the requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

V. Vogel's Approximation Method (VAM) [JNTU (MCA III) 2004]

Step 1. For each row of the transportation table identify the *smallest* and *next-to-smallest* cost. Determine the difference between them for each row. These are called *'penalties'*. Put them along side the transportation table by enclosing them in the parentheses against the respective rows. Similarly, compute these penalties for each column.

Step 2. Identify the row or column with the largest penalty among all the rows and columns. If a tie occurs, use any arbitrary tie breaking choice. Let the largest penalty correspond to ith row and let c_{ij} be the smallest cost in the ith row. Allocate the largest possible amount $x_{ij} = \min(a_i, b_j)$ in the cell (i, j) and cross-out the ith row or the jth column in the usual manner.

Step 3. Again compute the column and row penalties for the reduced transportation table and then go to *step* 2. Repeat the procedure until all the requirements are satisfied.

Note. In 1989, a new method for initial solution of transportation problem was developed by the Author of this book. The brief outlines of this new method are given in the Appendix. This method provides the initial solution very near to optimal solution. In most of the cases, the initial solution obtained by this new method proves to be optimum.

Q. 1. Explain with an example the North-West corner rule, the least cost method, and the Vogel's Approximation method for obtaining an initial basic feasible solution of a transportation problem. **[C.A. (Nov.) 91]**

2. Explain Vogel's Approximation Method of solving a transportation problem.

3. Explain the lowest cost entry method for obtaining an initial basic solution of a transportation problem. **[Madurai B.Sc. (Comp. Sc.) 92]**

4. List the various methods that can be used for obtaining an initial basic feasible solution for a transportation problem and describe any one of them. **[Garhwal M.Sc. (Math.) 95; Delhi B.Sc (Math.) 93]**

5. Discuss the algorithm of stepping stone method. **[VTU 2002]**

6. Write the steps to find the initial basic feasible solution by North-West corner rule. **[Bhubneshwar (IT) 2004]**

EXAMINATION PROBLEMS

1. Determine an initial basic feasible solution to the following transportation problem using the north-west corner rule, where O_i and D_j represent ith origin and jth destination respectively.

(i)

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	35

[IAS (Main) 89]

(ii)

		I	II	III	IV	Supply
	A	13	11	15	20	2,000
From	B	17	14	12	13	6,000
	C	18	18	15	12	7,000
Demand		3,000	3,000	4,000	5,000	

[Bharathidasan B.Sc (Math.) 90]

[**Ans.** (i) $x_{11}=6$, $x_{12}=8$, $x_{22}=2$, $x_{23}=14$, $x_{33}=1$, $x_{34}=4$, cost = Rs . 128
(ii) $x_{11}=2$, $x_{21}=1$, $x_{22}=3$, $x_{23}=2$, $x_{33}=2$, $x_{34}=5$]

(iii) To

							Available
	9	12	9	8	4	3	5
	7	3	6	8	9	4	8
From	4	5	6	8	10	14	6
	7	3	5	7	10	9	7
	2	3	8	10	2	4	3
Require	3	4	5	7	6	4	

[**Ans.** $x_{11}=3$, $x_{12}=2$, $x_{22}=2$, $x_{23}=5$, $x_{24}=1$, $x_{34}=6$, $x_{44}=0$, $x_{45}=6$, $x_{46}=1$, $x_{56}=3$.]

2. Determine an initial basic feasible solution to the following T.P. using the row/column minima method.

(i)

From	To A	B	C	Availability
I	50	30	220	1
II	90	45	170	4
III	250	200	50	4
Require	4	2	3	

(ii) To

	A	B	C	D	Available
I	6	3	5	4	22
From II	5	9	2	7	15
III	5	7	8	6	8
Demand	7	12	17	9	

[Bharathidasan B.Sc (Math.) 90]

[**Ans.** (i) $x_{12}=1$, $x_{21}=3$, $x_{22}=1$, $x_{31}=1$, $x_{33}=3$ (ii) $x_{12}=12$, $x_{13}=1$, $x_{14}=9$, $x_{23}=15$, $x_{31}=7$, $x_{33}=1$]

3. Obtain an initial basic feasible solution to the following T.P. using he matrix minima method.

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

where O_i and D_j denote ith origin and jth destination respectively.

[**Ans.** $x_{12}=6$, $x_{23}=2$, $x_{24}=6$, $x_{31}=4$, $x_{32}=0$, $x_{33}=6$]

4. Find the initial basic feasible solution of the transportation problem where cost-matrix is given below :

Destination

	A	B	*C*	*D*	Supply
I	1	5	3	3	34
Origin *II*	3	3	1	2	15
III	0	2	2	3	12
IV	2	7	2	4	19
Demand	21	25	17	17	

[**Ans.** $x_{11}=9$, $x_{12}=8$, $x_{14}=17$, $x_{23}=15$, $x_{31}=12$, $x_{42}=17$, $x_{43}=2$;
cost = Rs. 238 , using *'lowest cost entry method'*].

5. Determine an initial basic feasible solution using (i) Vogel's method and (ii) Row minima method, by considering the following transportation problem :

Destination

	1	2	3	4	Supply
1	21	16	15	13	11
Source 2	17	18	14	23	13
3	32	27	18	41	19
Demand	6	10	12	15	43

[JNTU (MCA III) 2004; VTU (BE Mech.) 2002; Gauhati (MCA) 92]

[**Ans.** $x_{14}=11$, $x_{21}=6$, $x_{22}=3$, $x_{24}=4$, $x_{32}=7$, $x_{33}=12$, cost = Rs. 686]

6. Determine an initial basic feasible solution to the following T.P. using : *(a)* North-west corner rule, and *(b)* Vogel's method.

		Destination				
	A_1	B_1	C_1	D_1	E_1	Supply
Origin A	2	11	10	3	7	4
B	1	4	7	2	1	8
C	3	9	4	8	12	9
Demand	3	3	4	5	6	21

[JNTU 2000]

[Ans. ***North west corner Rule*:** $x_{11} = 3$, $x_{12} = 1$, $x_{22} = 2$, $x_{23} = 4$, $x_{24} = 2$, $x_{34} = 3$, $x_{35} = 6$, cost = Rs. 153 .]
***Vogel's Method*:** $x_{14} = 4$, $x_{22} = 2$, $x_{35} = 6$, $x_{34} = 3$, $x_{32} = 1$, $x_{33} = 4$, $x_{34} = 1$, cost = Rs. 68].

7. Use *north west corner rule* to determine an initial basic feasible solution to the following T.P. when does it have a unique solution ?

	To			
	A	B	*C*	Supply
From *a*	2	7	4	5
b	3	3	1	8
c	5	4	7	7
d	1	6	2	14
Demand	7	9	18	34

[Meerut B.Sc. (Math.) 90]

Does the use of matrix minima method give a better (improved) basic feasible solution ? Why ?
[Ans. $x_{11} = 5$, $x_{21} = 2$, $x_{22} = 6$, $x_{32} = 3$, $x_{33} = 4$, $x_{43} = 14$, cost = Rs. 102; Yes]

8. Determine an initial basic feasible solution to the following T.P. using : (*a*) matrix minima method, (*b*) Vogel's approx. method.

	Destination				
	D_1	D_2	D_3	D_4	Supply
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
Demand	20	40	30	10	100

[Meerut 2002; IAS (Main) 88]

[Ans. For (*a*) and (*b*) bold : $x_{11} = 20$, $x_{13} = 10$, $x_{22} = 20$, $x_{23} = 20$, $x_{23} = 20$, $x_{24} = 10$, $x_{32} = 20$, cost = Rs. 180]

9. (i) Explain vogel's method by obtaining initial BFS of the following transportation problem :

	D_1	D_2	D_3	Supply
O_1	13	15	16	17
O_2	7	11	2	12
O_3	19	20	9	16
Demand	14	8	23	

[Ans. $x_{11} = 9$, $x_{12} = 3$, $x_{21} = 5$, $x_{23} = 7$, $x_{33} = 16$.]

	To			
	I	II	III	Supply
From A	50	30	200	1
B	90	45	170	3
C	250	200	50	4
Demand	4	2	2	

[Bharthidasan B.Sc (Math.) 90]

[Ans. $x_{11} = 1$, $x_{21} = 3$, $x_{31} = 0$, $x_{32} = 2$ $x_{33} = 2$]

10. Determine an initial feasible solution to the following T.P. using (a) (North-West corner rule, and (b) Vogel's approximation method :

Origin	Destination				Supply
	D_1	D_2	D_3	D_4	
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250	950

[Ans. $x_{11} = 200$, $x_{12} = 50$, $x_{22} = 175$, $x_{24} = 125$, $x_{33} = 275$, $x_{34} = 125$.] **[Bhubneshwar (IT) 2004]**

11. Determine the initial bfs to the following TP by matrix method.

		To			
		A	B	C	a_i
From	I	14	15	10	20
	II	21	13	19	24
	III	17	26	9	12
	b_j	28	22	6	56

[AIMS Banglore (MBA) 2002]

Optimum Solution

11.9. MOVING TOWARDS OPTIMALITY

After obtaining an initial basic feasible solution to a given transportation problem, the next question is 'how to arrive at the optimum solution'. The basic steps for reaching the optimum solution are the same as given for simplex method, namely :

Step 1. Examination of initial basic feasible solution for *non-degeneracy.* If it is *degenerate,* some modification is required to make it non-degenerate (as discussed in ***Sec.* 11.11**).

Step 2. *(i)* Determination of net-evaluations (cost-difference) for empty cells.
(ii) Optimality test of current solution.

Step 3. Selection of the entering variable, provided *Step 2(ii)* indicates that the current solution can be improved.

Step 4. Selection of the leaving variable.

Step 5. Finally, repeating the ***steps 1 through 4*** until an optimum solution is obtained.

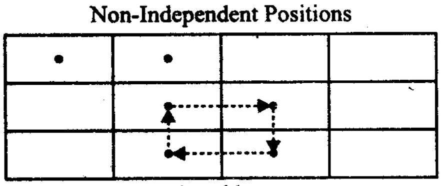
11·9-1. To Examine the Initial Basic Feasible Solution for Non-degeneracy.

A basic feasible solution of an $m \times n$ transportation problem is said to be *non-degenerate,* if it has the following two properties :

(1) *Initial BFS must contain exactly $m + n - 1$ number of individual allocations.*
For example, in 3×4 transportation problem, the number of individual allocations in BFS obtained by any one of the methods discussed so far is equal to 6, *i.e.*, $3 + 4 - 1$, which can be easily verified from ***Tables* 11·5, 11·12, 11·13 and 11·19.**

(2) *These allocations must be in 'independent positions.'*
Independent positions of a set of allocations mean that it is always impossible to form any closed loop through these allocations. ***Tables* 11·20, 11·21** show the non-independent, and **11·23** independent positions by '•'.

Table 11·20
Non-Independent Positions



closed loop

Table 11·21
Non-Independent Positions

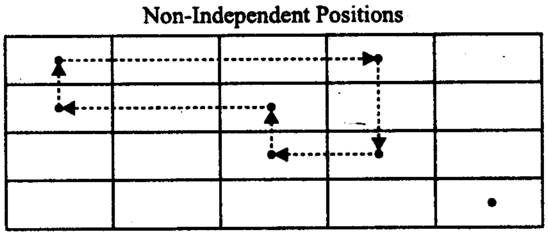


Table 11·22
Non-Independent Positions

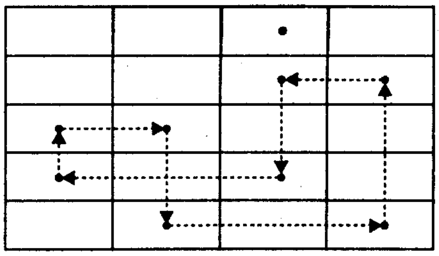


Table 11·23
Independent Positions

•				•
			•	•
		•		•

In above allocation patterns of different problems, the dotted lines constitute what are known as ***loops.*** A loop may or may not involve all allocations. It consists of (at least 4) horizontal and vertical lines with an allocation at each corner which, in turn, is a join of a horizontal and vertical lines. At this stage, loop of ***Table 11·22*** should be particularly noted. Here two lines intersect each other at cell (4, 2) and do not simply join ; therefore this is not to be regarded as a corner. Such allocations in which a loop can be formed are known as non-independent positions whereas those (of ***Table 11·23***) in which a loop cannot be formed are regarded as independent.

11·9-2 . Determination of Net-Evaluations (u, v method).

Unlike the simplex method, the net-evaluations for a transportation problem can be determined more easily by using the properties of the *primal* and *dual* problems.

Let us consider the following m-origin, n-destination transportation problem :

Determine x_{ij} so as to minimize $z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\,(c_{ij})$, subject to the constraints :

$$\sum_{j=1}^{n} x_{ij} = a_i \quad \text{or} \quad a_i - \sum_{j=1}^{n} x_{ij} = 0\,, \qquad \text{for } i = 1, 2, \dots, m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \text{or} \quad b_j - \sum_{i=1}^{m} x_{ij} = 0\,, \qquad \text{for } j = 1, 2, \dots, n$$

and $\qquad x_{ij} \geq 0$, for all i and j .

Let $u_1, u_2, \dots u_m$ and $v_1, v_2, \dots v_n$ be the dual variables associated with the above *origin* and *destination* constraints, respectively. Since the above *primal* T.P. has $m+n$ constraint equations in mn number of variables, so the *dual* of above problem will contain mn constraints in $m+n$ dual variables in the form :

$$u_i + v_j \leq c_{ij} \text{ and } u_i, v_j \text{ are unrestricted for all } i \text{ and } j \quad (\because \text{ constraints in the primal are equations})$$

From ***chapter 7,*** we recall that for any standard primal L.P.P. with basis **B** and associated cost vector $\mathbf{C_B}$, the associated solution to its dual is given by $\mathbf{W_B} = \mathbf{C_B}\,\mathbf{B}^{-1}$. Thus, if $\mathbf{a}_j$ is the jth column of the primal constraint matrix, then an expression for evaluating the net-evaluation for minimization problem is given by

$$c_j - z_j = c_j - \mathbf{C_B}\,(\mathbf{B}^{-1}\,\mathbf{a}_j) = c_j - \mathbf{W_B a_j} \qquad \text{for all } j\,.$$

But, in the present case of *rectangular* transportation problem (which is the special case of L.P.P.), the dual solution can be represented by

$$(\mathbf{u}, \mathbf{v}) = (u_1, \dots, u_m, v_1, \dots, v_n)$$

and therefore the net evaluations are analogously obtained by simply replacing $c_j \to c_{ij}$, $\mathbf{W_B} \to (\mathbf{u}, \mathbf{v})$, $\mathbf{a}_j \to \mathbf{a}_{ij}$ in the above formula to get

$$c_{ij} - z_{ij} = c_{ij} - (\mathbf{u}, \mathbf{v})\,\mathbf{a}_{ij} = c_{ij} - (u_1, \dots, u_m, v_1, \dots, v_n)\,[\mathbf{e}_i + \mathbf{e}_{m+j}]$$

$$= c_{ij} - (u_i + v_j)\ ;\ i = 1, \dots, m\ ;\ j = 1, \dots, n\,,$$

where $\mathbf{a}_{ij}$ is the column vector of the constraint matrix associated with the *rectangular* variable x_{ij} . For simplicity, we shall denote the net evaluation $c_{ij} - z_{ij}$ by d_{ij} in all further discussions.

Now, since the net evaluations must vanish for the basic variables it follows that $d_{ij} = c_{ij} - (u_i + v_j)$ for all non-basic cells (i, j) where u_i and v_j satisfy the relation $c_{rs} = u_r + v_s$ for all basic cells (r, s) . Except for the degeneracy case, there are $m + n - 1$ dual equations in $m + n$ dual unknowns for the $m + n - 1$ basic cells. We can arbitrarily assign the value to one of these unknowns u_r and v_s and solve uniquely for the remaining $m+n-1$ variables. After this arbitrary assignment, say $u_1 = 0$, the rest of the values are obtained by simple addition and subtraction. Once we determine all the u_i and v_j, the net evaluations for all the non-basic cells are easily determined by the relation $d_{ij} = c_{ij} - (u_i + v_j)$.

Alternative Method to Determine Net-Evaluations.

The necessary condition for optimality can also be established in the form of the following theorem.

Theorem 11.7. *If we have a feasible solution consisting of $m+n-1$ independent allocations, and if numbers u_i and v_j satisfying $c_{rs} = u_r + v_s$, for each **occupied** cell (r, s), then the evaluation d_{ij} corresponding to each **empty** cell (i, j), is given by $d_{ij} = c_{ij} - (u_i + v_j)$.*

Proof. The transportation problem is to find $x_{ij} \geq 0$ in order to minimize

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\,(c_{ij}) \qquad \text{...(11·6)}$$

subject to the restrictions

$$\sum_{j=1}^{n} x_{ij} = a_i\,,\; i = 1, 2, 3, \ldots, m \qquad \text{...(11·7)}$$

$$\sum_{i=1}^{m} x_{ij} = b_j\,,\; j = 1, 2, 3, \ldots, n \qquad \text{...(11·8)}$$

and $\qquad x_{ij} \geq 0$ for all i and j.

The restrictions (11·7) and (11·8) may be written as

$$0 = a_i - \sum_{j=1}^{n} x_{ij}\,,\; i = 1, 2, 3, \ldots, m \qquad \text{...(11·9)}$$

$$0 = b_j - \sum_{i=1}^{m} x_{ij}\,,\; j = 1, 2, 3, \ldots, n\,. \qquad \text{...(11·10)}$$

Any multiple of each of these restrictions [(11·9) and (11·10)] can be legally added to the objective function (11·6) to try to eliminate the basic variable. These multiples are denoted by u_i $(i = 1, 2, \ldots, m)$ and v_j $(j = 1, 2, \ldots, n)$, respectively. Thus,

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}\,(c_{ij}) + \sum_{i=1}^{m} u_i\,(a_i - \sum_{j=1}^{n} x_{ij}) + \sum_{j=1}^{n} v_j\,(b_j - \sum_{i=1}^{m} x_{ij}) \qquad \text{...(11·11a)}$$

or

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} [c_{ij} - (u_i + v_j)]\, x_{ij} + \sum_{i=1}^{m} u_i\, a_i + \sum_{j=1}^{n} v_j\, b_j\,. \qquad \text{...(11·11b)}$$

The necessary condition for a coefficient of zero is

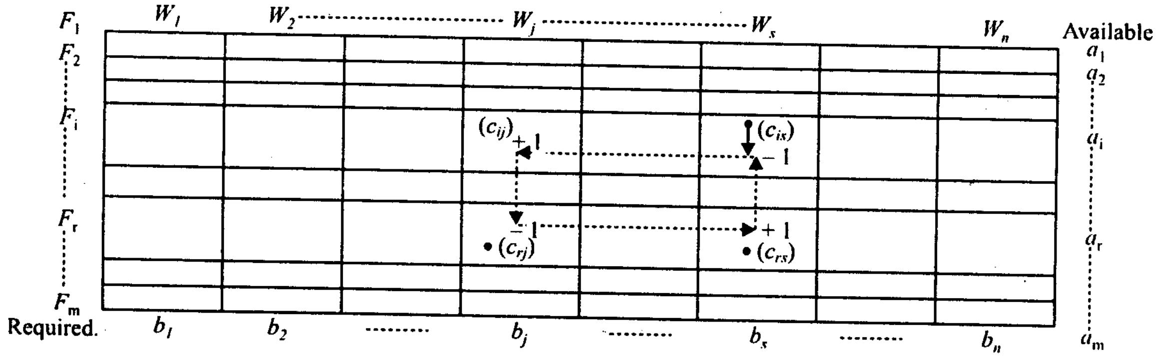
$$c_{rs} = u_r + v_s \qquad \text{...(11·12)}$$

for each basic variable x_{rs}, *i.e.*, for each occupied cell (r, s). Since there are $m+n-1$ number of equations of the form (11·12) in $(m+n)$ number of unknowns (u_i and v_j), so if assignment is made to an arbitrary value of one of the u_i or v_j, then rest of the $(m+n-1)$ unknowns can be easily solved algebraically. One reasonable and convenient rule, which will be adopted here, is to select the u_i which has the largest number of allocations in its row, and assign it the value of zero. And $c_{ij} = u_i + v_j$ immediately yields v_j for columns containing those allocations.

To prove the required result, first suppose that the empty cell (i, j) be connected to occupied cells by a closed loop (*Table 11·24*).

First, allocate + 1 unit to the empty cell (i, j), and in order to balance the total requirement of warehouse W_j, add – 1 unit to occupied cell (r, j). Consequently, the total amount available from factory F_r will be balanced by adding + 1 unit to occupied cell (r, s), which in turn causes column W_s to become unbalanced. So balance the column W_s by adding – 1 unit to the occupied cell (i, s).

Table 11·24



This process will give the *cost difference* d_{ij} [called the empty *cell evaluation* for (i, j)] between the new solution and the original solution.

Thus, $$d_{ij} = c_{ij} - c_{rj} + c_{rs} - c_{is} \, . \qquad ...(11\cdot13)$$

Using the result (11·12) for all occupied cells such as (r, j), (r, s) and (i, s),

$$d_{ij} = c_{ij} - (u_r + v_j) + (u_r + v_s) - (u_i + v_s) = c_{ij} - (u_i + v_j) \, . \qquad ...(11\cdot14)$$

This proves the result for a loop of square shape connecting the empty cell(i, j) to occupied cells. In a similar fashion, generalize the empty cell (i, j) to occupied cells.

This completes the proof of the theorem.

11·9-3. The Optimality Test

If the cost difference $d_{ij} \geq 0$ (which implies increase in cost for each empty cell), then the BFS under test must be optimal. Otherwise, if $d_{ij} < 0$ (because negative difference implies decrease in cost) for one or more empty cells, then it would be better to reduce the cost more by allocating as much as possible to the cell with the largest negative (smallest) value of d_{ij}. This way, it is possible to improve the BFS successively for reduced cost till the optimal solution is obtained for which $d_{ij} \geq 0$ for each empty cell.

The optimality test for given BFS of the transportation problem may be summarized as follows :

1. ***Start with a basic feasible solution consisting of $m + n - 1$ allocations in independent positions.***
2. ***Determine a set of $m + n$ numbers u_i ($i = 1, 2, 3, \ldots, m$) and v_j ($j = 1, 2, 3, \ldots, n$) such that for each occupied cell (r, s) $c_{rs} = u_r + v_s$.***
3. ***Calculate cell evaluations (unit cost differences) d_{ij} for each empty cell (i, j) by using the formula***
$$d_{ij} = c_{ij} - (u_i + v_j) \, .$$
4. ***Finally, examine the matrix of cell evaluations d_{ij} for negative entries and conclude that –***
   - *(i)* ***solution under test is optimal, if none is negative ;***
   - *(ii)* ***alternative optimal solutions exist, if none is negative but any is zero ;***
   - *(iii)* ***solution under test is not optimal, if any is negative, then further improvement is required by repeating the above process.***

We now proceed to answer the question : how to improve the current BFS if it is not optimal, *i.e.* if all $d_{ij} \leq 0$.

11·9-4. Selection of Entering Variable

Here our aim is to minimize the cost of transportation. So the current basic feasible solution will not be optimum so long as any of the net evaluation d_{ij} is negative. Thus if all d_{ij} are non-negative, the current solution is an optimum one, otherwise using simplex like criterion we select such variable x_{rs} to enter the basis for which the net evaluaton $d_{rs} = \min_{i,j} \{d_{ij} < 0\}$.

11·9-5. Selection of Leaving Variable

Our next step will be to determine the leaving basic variable and then to determine the new improved basic solution. The simplex like leaving criterion in the notations of transportation problem states that if the variable x_{rs} is selected to enter the basis, then the basic variable x_{Bi} corresponding to the *minimum ratio* :

$\min \left\{ \frac{x_{Bi}}{y_{irs}}, y_{irs} > 0 \right\}$, will leave the basis. However, due to special structure of the transportation problem the above criterion has been simplified to a great extent.

Theorem 11.8. *Let* $\{\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_{m+n-1}\}$ *be a basis set for the column vectors of the coefficient matrix* A *of m-origin, n-destination transportation problem. In the representation of any non-basic* $\mathbf{a}_{rs}$ *as a linear combination:*

$$\mathbf{a}_{rs} = \sum_{i=1}^{m+n-1} y_{irs} \mathbf{b}_i ,$$

of basis vectors, every scalar element (y_{irs}) *is either* -1 or $+1$.

Proof. Since $\{\mathbf{a}_{rs}, \mathbf{b}_1, \ldots, \mathbf{b}_{m+n-1}\}$ is a linearly dependent set (by *Theorem 11·6*), the set of the associated cells contains a loop. So the cell (r, s) must be in a loop. The vectors $\mathbf{b}_i$'s are, of course, some column vectors $\mathbf{a}_{ij}$'s of A.

Suppose the set of associated cells contains the following loop :

$$L = \{(r, s), (r, t), (p, t), (p, q), \ldots, (u, v), (u, s)\},$$

where $\mathbf{a}_{rs}, \mathbf{a}_{rt}, \ldots, \mathbf{a}_{us}$ are the given basis vectors ($\leq m + n - 1$).

Now, since $\mathbf{a}_{ij} = \mathbf{e}_i + \mathbf{e}_{m+j}$ for all i and j, and because the number of cells in a loop is always even, we have

$$\mathbf{a}_{rs} - \mathbf{a}_{rt} + \mathbf{a}_{pt} - \mathbf{a}_{pq} + \ldots + \mathbf{a}_{uv} - \mathbf{a}_{us} = \mathbf{0}$$

which yields, $\mathbf{a}_{rs} = \mathbf{a}_{rt} - \mathbf{a}_{pt} + \mathbf{a}_{pq} - \ldots - \mathbf{a}_{uv} + \mathbf{a}_{us}$.

This is the unique representations of $\mathbf{a}_{rs}$ as a linear combination of basis vectors; and hence the y_{irs} elements associated with the basis vector in the above representation are $+1, -1, \ldots, -1$ and $+1$.

This completes the proof of the theorem.

Thus if x_{rs} is the entering variable, the basic variable x_{Bt} will leave the basis if $x_{Bt} = \min \{x_{Bi}\}$, since positive y_{irs} is $+1$ for all basic variables.

11·9-6. Determination of New (Improved) Basic Feasible Solution

After the entering and leaving variables are determined, all that remains to determine is ***'the new (improved) basic solution'***. The usual transformation formulae for obtaining the new basic solution, in the simplex transportation notation, are given by

$$\hat{x}_{Bt} = \frac{x_{Bt}}{y_{trs}}, \quad y_{trs} > 0 \qquad \text{and} \qquad \hat{x}_{Bi} = x_{Bi} - \frac{x_{Bt}}{y_{trs}} y_{irs} \text{ for all } i \neq t$$

Since $y_{trs} = +1$, $y_{irs} = \pm 1$ for basic variables, and $y_{irs} = 0$ for non-basic variables, the above transformations get simplified to

$$\hat{x}_{Bt} = x_{Bt},$$
$$\hat{x}_{Bi} = x_{Bi} - y_{irs} x_{Bt} = x_{Bi} \pm x_{Bt} \text{ for all } x_{Bi} \in X, i \neq t$$
and $$\hat{x}_{Bi} = x_{Bi}, \quad \text{for all } x_{Bi} \notin X,$$

where X is the set of those basic variables whose corresponding cells are included in the loop as identified in the preceding theorem.

In practice, however, this improvement procedure is quite simple. The working-rule is outlined in the following steps :

11·9-7. Working Rule to Obtain Leaving Variable and Improved Basic Feasible Solution

Step 1. After identifying the entering variable x_{rs}, describe a loop which starts and ends at the non-basic cell (r, s) connecting only the basic cells. *Such a closed path exists and is unique for any non-degenerate basic solution.*

Step 2. The amount (say θ) to be allocated to the entering variable is interchangably *subtracted from* and *added to* the successive end points of the closed loop so that the supply and demand constraints always remain satisfied.

Step 3. Then the minimum value of θ, which will render non-negative values for all the basic variables in the new solution, is obtained. This consequently, determines the leaving variable.

For illustration, see ***Example* 2** in ***Sec* 11·10–2**.

Q. 1. Describe any two methods of constructing a basic feasible solution for the transportation problem. How would you test for optimality of a given solution for the transportation problem ? **[Meerut (Stat.) 92]**

2. Show that the transportation and assignment problem can be regarded as the particular cases of L.P.P. **[Meerut (Stat.) 98]**

11.10. TRANSPORTATION ALGORITHM FOR MINIMIZATION PROBLEM

The transportation algorithm for minimization problem can be summarized in the following steps.

The Algorithm :

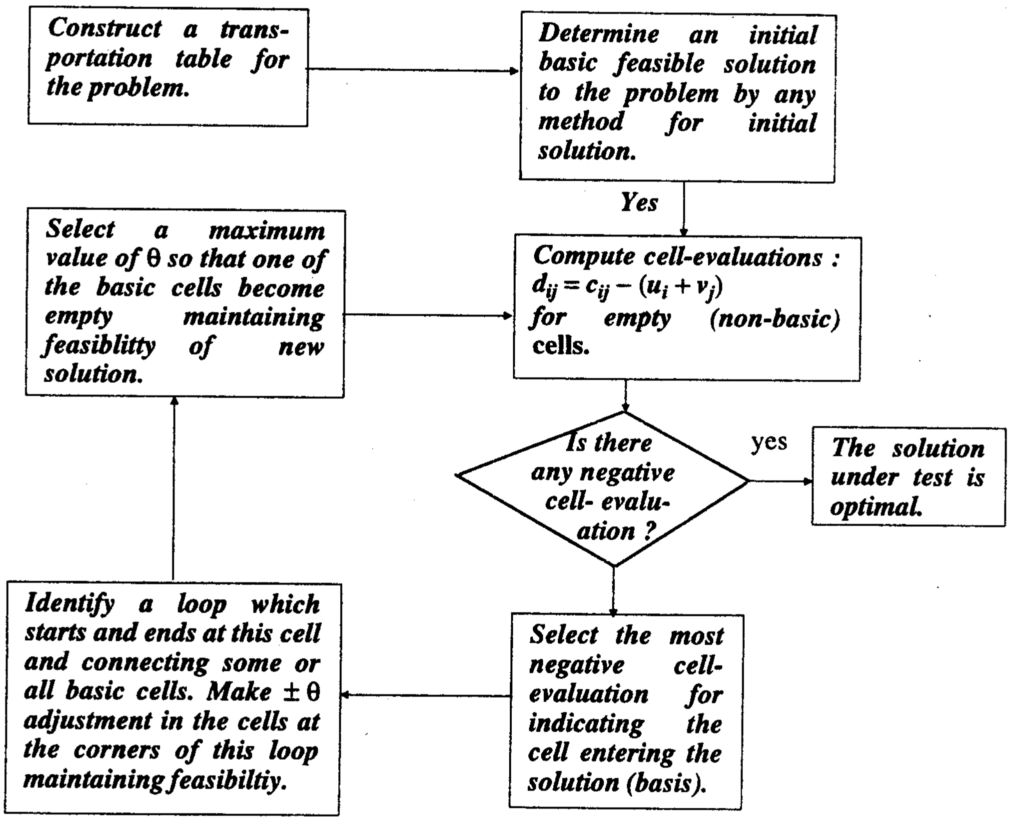
Step 1. First, construct a transportation table entering the origin capacities a_i, the destination requirements b_j and the costs c_{ij}, as shown in *Table 11.14* (page 272)

Step 2. Find an initial basic feasible solution by Vogel's method or by any of the given methods. Enter the solution at the centre (•) of basic cells.

Step 3. For all the basic variables x_{ij}, solve the system of equations $u_i + v_j = c_{ij}$, for all i, j for which cell (i, j) is in the basis, starting initially with some $u_i = 0$ and entering successively the values of u_i and v_j on the transportation table as shown in *Table 11.26.* (page 285)

Step 4. Compute the cost differences $d_{ij} = c_{ij} - (u_i + v_j)$ for all the non-basic cells and enter them in the upper right corners of the corresponding cells.

Flowchart of Transportation Algorithim



Step 5. Apply optimality test by examining the sign of each d_{ij} :

(*i*) If all $d_{ij} \geq 0$, the current basic feasible solution is an optimum one.

(*ii*) If at least one $d_{ij} < 0$ (negative), select the variable x_{rs} (having the most negative d_{rs}) to enter the basis.

Step 6. Let the variable x_{rs} enter the basis. Allocate an unknown quantity say θ , to the cell (r, s) . Then construct a loop that starts and ends at the cell (r, s) and connects some of the basic cells. The amount θ is added to and subtracted from the transition cells of the loop in such a manner that the availabilities and requirements remain satisfied.

Step 7. Assign the largest possible value to θ in such a way that the value of at least one basic variable becomes zero and other basic variables remain non-negative (≥ 0) . The basic cell whose allocation has been made zero will leave the basis.

Step 8. Now, return to ***step 3*** and then repeat the process until an optimum basic feasible solution is obtained.

The above iterative procedure determines an optimum solution in a finite number of steps. This method is called *MODI METHOD*. and can be easily remembered with the help of the following FLOW-CHART.

Q. 1. Give an algorithm for solving transportation problem.

2. State the transportation problem. Describe clearly the steps involved in solving the problem.
3. Describe the transportation problem. Give a method of finding an initial feasible solution. Explain what is meant by an optimality test ? Give the method of improving over the initial solution to reach the optimal feasible solution. **[Meerut 94]**
4. Assume that in a transportation problem the demand and supply levels are all positive and integral. Show that there exists an integral optimal solution if the total demand equals total supply levels.
5. Describe the computational procedure of optimality test in a transportation problem.
6. Explain briefly the step-wise description of the computational procedure for solving the transportation problem. **[Delhi B.Sc. (Math.) 91]**
7. Develop methematical model of a balanced transportation problem. Prove that it always has a feasible solution. **[IAS (Main) 99]**
8. How do you diagnose that the given transportation problem is having more than one optimal alternate optimal solution. **[AIMS (Ind.) 2002]**

11·10–1. Computational Demonstration of Optimality Test

Example 3. (a) *Obtain an initial basic feasible solution to the transportation problem of* ***Example 1.*** *Is this solution an optimal solution ? If not, obtain the optimal solution.*

[IGNOU 2001; JNTU (Mach.) 99; Gauhati (MCA) 91]

(b) *If a company is spending Rs. 1000 on transportation of its units to four warehouses from three factories. What can be the maximum saving by optimal scheduling.*

Solution. (a) Computational demonstration for optimality is performed by taking the initial basic feasible solution of ***Example 1*** with $m + n - 1$ allocations in independent positions with transportation cost of Rs 779 obtained (by *Vogel's Method*). This initial basic feasible solution is given in *Table 11·25*.

Table 11·25

	W_1	W_2	W_3	W_4	Available
F_1	5(19)			2(10)	7
F_2			7(40)	2(60)	9
F_3		8(8)		10(20)	18
Required	5	8	7	14	

Step 1. The initial BFS has $m + n - 1$ allocations, that is, $3 + 4 - 1 = 6$ allocations in independent positions. Therefore, condition (1) of optimality test [in sec. *11·9–3*] is satisfied.

Step 2. Since u_i ($i = 1, 2, 3$) and v_j ($j = 1, 2, 3, 4$) are to be determined by means of unit cost in the respective occupied cells only, assign a u-value of any particular amount (conveniently *zero*) to any particular row (convenient rule is to select the u_i which has the largest number of allocations in its row). Since all rows contain the same number of allocations, take any of the u_i (say u_3) equal to zero.

When $u_3 = 0$, $v_4 = 20$ (since $c_{34} = u_3 + v_4$; $c_{34} = 20$). Similarly, $c_{32} = u_3 + v_2$ or $8 = 0 + v_2$ or $v_2 = 8$. Again, $c_{14} = u_1 + v_4$ or $10 = u_1 + 20$ (since $c_{14} = 10$), then $u_1 = -10$. In the same way :$60 = 20 + u_2$, which gives $u_2 = 40$; $19 = u_1 + v_1$ or $19 = -10 + v_1$, which gives $v_1 = 29$; $40 = u_2 + v_3$ or $40 = 40 + v_3$, which gives $v_3 = 0$. This completes the set of u_i ($i = 1, 2, 3$) and v_j ($j = 1, 2, 3, 4$) as shown in *Table 11·26*.

Table 11·26

				u_i
• (19)			• (10)	-10
		• (40)	• (60)	40
	• (8)		• (20)	0
v_j 29	8	0	20	

Step 3. To compute the matrix of cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ for *empty cells*, it is convenient to write a matrix $[c_{ij}]$ for *empty cells* and the matrix of numbers $[u_i + v_j]$ for *empty cells* only, then subtract the later matrix from the former one.

Table 11·27 ***(from Table 11·3)***
Matrix $[c_{ij}]$ for empty cells

•	(30)	(50)	•
(70)	(30)	•	•
(40)	•	(70)	•

Table 11·28
Matrix $[u_i + v_j]$ for empty cells

•	−2	−10	•
69	48	•	•
29	•	0	•

Now, subtracting the matrix $[u_i + v_j]$ from the matrix $[c_{ij}]$, *i.e.*, (*Table 11·27 – Table 11·28*), the following matrix $[c_{ij} - (u_i + v_j)]$ of cell evaluations is obtained.

Table 11·29 gives the empty cell evaluations : $d_{12} = 32$, $d_{13} = 60$, $d_{21} = 1$, $d_{22} = -18$, $d_{31} = 11$ and $d_{33} = 70$. The largest negative cell evalua- tion (marked √) is $d_{22} = -18$ which indicates that allocation of one unit to this empty cell (2, 2) will reduce the achieved cost of Rs 779 by Rs. 18. So allocate (say, θ) to cell (2, 2) as much as possible, followed by alternately subtracting and adding the amount of this allocation to other corners of the loop in order to restore feasibility (non-negativity of allocations). For this purpose, the initial basic feasible solution can be read from *Table 11·30*. It is easily seen by the following rule that at the most θ = 2 units can be allocated from cell (2, 4) to cell (2, 2) still satisfying the row and column total and non-negativity restrictions on the allocations.

Table 11·29

•	32	60	•
1	−18	•	•
11	•	70	•

Table 11.30

				Available
5 • (19)			2 • (10)	7
	+θ √	7 • (40)	2 − θ • (60)	9
	8 − θ • (8)		10 + θ • (20)	18
Required 5	8	7	14	

A Rule to Determine θ : Reallocation is done by transfering the maximum possible amount θ in the marked (√) cell. The value of θ, in general, is obtained by equating to zero the minimum of the allocations containing − θ (not + θ) at the corners of the closed loop. That is, in *Table 11·30*, min $[8 - \theta, 2 - \theta] = 0$ or $2 - \theta = 0$ or $\theta = 2$ units. Thus improved basic feasible solution is given in *Table 11·31*.

Table 11·31

				Available
5(19)			2(10)	7
	2(30)	7(40)		9
	6(8)		12(20)	18
Required 5	8	7	14	

The cost for this solution becomes

$$= 5\,(19) + 2\,(10) + 2\,(30) + 7\,(40) + 6\,(8) + 12\,(20) = \text{Rs } 743.$$

The cost of Rs 743 is Rs ($2 \times 18 = 36$) less than Rs 779 which was expected also.

Step 4. Test this improved solution (*Table 11·31*) for optimality by repeating *steps 1, 2* and *3*. In each step, following matrices are obtained.

Table 11·32

Matrix $[c_{ij}]$ for empty cells

•	(30)	(50)	
(70)	•	•	(60)
(40)	•	(70)	•

Table 11·33

Matrix for u_i and v_j

					u_i
	• (19)			• (10)	−10
		• (30)	• (40)		22
		• (8)		• (20)	0
$v_j \rightarrow$	29	8	18	20	

Table 11·34

Matrix $[u_i + v_j]$ for empty cells

•	−2	8	•
51	•	•	42
29	•	18	•

Table 11·35

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

•	32	42	•
19	•	•	18
11	•	52	•

Since none of the cell evaluations is negative, *i.e.*, $d_{12} = 32$, $d_{13} = 42$, $d_{21} = 19$, $d_{24} = 18$, $d_{31} = 11$ and $d_{33} = 52$, the solution given in ***Table 11·31*** is optimal with minimum cost of Rs. 743.

(b) Maximum saving = Rs. 1000 − Rs. 743 = Rs. 257 . **Ans.**

11·10-2. More Solved Examples

Example 4. *Solve the following transportation problem in which cell entries represent unit costs.*

Table 11·36

	To			Available
From	2	7	4	5
	3	3	1	8
	5	4	7	7
	1	6	2	14
Required	7	9	18	34

Solution. By *Vogel's method*, the following initial basic feasible solution having the transportation cost of Rs. 80 is obtained.
To test the solution for optimality, required tables are given below.

Table 11·37

				Available
	5(2)			5
			8(1)	8
		7(4)		7
	2(1)	2(6)	10(2)	14
Required	7	9	18	

Table 11·38

[Matrix for set of u_i and v_j]

				u_i
	(2)			1
			(1)	−1
		(4)		−2
	(1)	(6)	(2)	0
v_j	1	6	2	

Table 11·39

[Matrix c_{ij} for empty cells]

•	(7)	(4)
(3)	(3)	•
(5)	•	(7)
•	•	•

Table 11·40

[Matrix $(u_i + v_j)$ for empty cells]

•	7	3
0	5	•
−1	•	0
•	•	•

Table 11·41

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

•	0	1
3	−2 √	•
6	•	7
•	•	•

From *Table* 11·41, it is observed that the cell evaluation, $d_{22} = -2$, is negative. Therefore, the solution [*Table* 11·37] under test is not optimal.

The solution can be improved as shown in ***Tables 11·42*** and ***11·43***.

Table 11·42

5 (2)			5
	√+θ	8 − θ (1)	8
	7 (4)		7
2 (1)	2 − θ (6)	10 + θ (2)	14
7	9	18	

Table 11·43

5(2)			5
	2(3)	6(1)	8
	7(4)		7
2(1)		12(2)	14
7	9	18	

Here min $[8-\theta, 2-\theta] = 0$ or $2-\theta = 0$ or $\theta = 2$ units.

Table 11·44

Matrix for set of u_i and v_j

				u_i
	(2)			1
		(3)	(1)	−1
		(4)		0
	(1)		(2)	0
v_j	1	4	2	

Table 11·45

Matrix $[c_{ij}]$ for empty cells

•	(7)	(4)
(3)	•	•
(5)	•	(7)
•	(6)	•

Table 11·46

Matrix $[u_i + v_j]$ for empty cells

•	5	3
0	•	•
1	•	2
•	4	•

Table 11·47

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

•	2	1
3	•	•
4	•	5
•	2	•

Since all empty cell evaluations in ***Table* 11·47** are positive; the solution in *Table 11·43* is optimal.

The minimum cost for this solution is = 5(2) + 2(3) + 6(1) + 7(4) + 2(1) + 12(2) = Rs. 76.

Example 5. *Solve the transportation problem* :

	D_1	D_2	D_3	D_4	Available
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
Required	20	40	30	10	100 Total

Solution. By ***'Lowest Cost Entry Method'***, the starting basic feasible solution is obtained having the transportation cost Rs 180.

Table 11·48

					Available
	20(1)		10(1)		30
		20(3)	20(2)	10(1)	50
		20(2)			20
Requirement	20	40	30	10	

To test this solution for optimality, following tables are obtained :

Table 11·49

Matrix for set of u_i and v_j

				u_i
(1)		(1)		−1
	(3)	(2)	(1)	0
	(2)			−1
v_j: 2	3	2	1	

Table 11·50

Matrix $[c_{ij}]$ for empty cells only

•	(2)	•	(4)
(3)	•	•	•
(4)	•	(5)	(9)

Table 11·51

Matrix $(u_i + v_j)$ for empty cells

•	2		0
2	•	•	•
1	•	1	0

Table 11·52

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

•	0	•	4
1	•	•	•
3	•	4	9

Table 11·52 shows that all empty cell evaluations are non-negative. Hence the solution under test is optimal.

Furthermore, the cell evaluation $d_{12} = 0$ indicates that alternative optimal solutions also exist.

Example 6. *Determine the optimum basic feasible solution to the following transportation problem.*

To

	A	B	C	Available
I	50	30	220	1
	90	45	170	3
III	250	200	50	4
Required	4	2	2	

[I.A.S. (Main) 91]

Solution. The intial basic feasible solution can be easily obtained by two different methods as follows :

By Lowest Cost Method

	1 (30)		1
2 (90)	1 (45)		3
2 (250)		2 (50)	4
4	2	2	

Cost $z = 1 \times 30 + 2 \times 90 + 1 \times 45 + 2 \times 250 + 2 \times 50$
= Rs. 855.

By Vogel's Method

1 (50)	(30)	220	1
3 (90)	(45)	(170)	3
(250)	2 (200)	2 (50)	4
4	2	2	

Cost $z = 1\ (50) + 3\ (90) + 2\ (200) + 2\ (50)$
= Rs. 820.

Proceeding to test the initial solution (by *lowest cost method)* for optimality, following tables are obtained :

Matrix for u_i and v_j

			u_i
	(30)		−15
(90)	(45)		0
(250)		(50)	160
v_j: 90	45	−110	

Matrix $[c_{ij}]$ for empty cells

(50)	•	(220)
•	•	(170)
•	(200)	•

Matrix $(u_i + v_j)$ for empty cells

			u_i
75	•	−125	−15
•	•	−110	0
•	205	•	160
v_j: 90	45	−110	

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

−25	•	345
•	•	280
•	−5	•

Since cell evaluations $d_{ij} = c_{ij} - (u_i + v_j)$ are not all non-negative, the solution under test is not optimal.

First Iteration. Since the most negative cell evaluation is $d_{11} = -25$, so allocate as much as possible to cell (1, 1). This necessitates the shifting of 1 unit to this cell (1, 1) from cell (1, 2) as directed by the closed loop in the following table. Thus, cell (1, 1) enters the solution, while cell (1, 2) leaves the solution, *i.e.*, it becomes empty.

Table 11·53

Here min $[1 - \theta, 2 - \theta] = 0$ or $1 - \theta = 0$ or $\theta = 1$ unit.

$+\theta$ √	$1 - \theta$ (30)		1
$2 - \theta$ (90)	$1 + \theta$ (45)		3
2 • (250)		2 • (50)	4
4	2	2	

Improved solution (z = Rs. 830)

1(50)			1
1(90)	2(45)		3
2(250)		2(50)	4
4	2	2	

To test the improved solution for optimality :

Matrix for u_i and v_j

			u_i
(50)			50
(90)	(45)		90
(250)		(50)	250
v_j 0	−45	−200	

Matrix $[c_{ij}]$ for empty cells

•	(30)	(220)
•	•	(170)
•	(200)	•

Matrix $(u_i + v_j)$ for empty cells

			u_i
•	5	−150	50
•	•	−110	90
•	205	•	250
v_j 0	−45	−200	

Matrix for $[c_{ij} - (u_i + v_j)]$

•	25	370
•	•	280
•	−5 (√)	•

Since the cell evaluation, $d_{32} = -5$ (negative), the improved solution under test is not optimal. So proceed to second iteration.

Second Iteration. Since the largest negative cell evaluation is $d_{32} = -5$, so allocate as much as possible to cell (3, 2). Thus, the cell (3, 1) leaves the solution while (3, 2) enters the solution. Cell (2, 2) may also be selected for leaving.

Here $2 - \theta = 0$ or $\theta = 2$ units.

1 • (50)			1
$1 + \theta$ (90)	$2 - \theta$ (45)		3
$2 - \theta$ (250)	$+\theta$ √		4
4	2	2	

Improved solution (z = Rs. 820)

1(50)			1
3(90)	0(45)		3
	2(200)	2(50)	4
4	2	2	

Again, proceed as earlier to test the next improved solution for optimality. It has been observed that all cell evaluations are now non-negative. Hence the solution under test is optimal.

$$z = 1 \times 50 + 3 \times 90 + 0 \times 45 + 2 \times 200 + 2 \times 50 = \text{Rs. } 820.$$

This solution was initially obtained by *Vogel's method.*

Example 7. *Determine the optimum basic feasible solution to the following transportation problem* :

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	24 (Total)

where O_i and D_j *denote* ith *origin and* jth *destination, respectively.*

Solution. The initial solution to this problem can be easily obtained. In order to make the number of allocations equal to $m+n-1$, allocate 0 amount to cell (3, 2), otherwise it is not possible to test for optimality Also see **sec 11.11** on ***degeneracy*** for handling such situations. $z = 12 + 4 + 0 + 12 + 0 + 0 =$ Rs. 28.

	6(2)			6
		2(2)	6(0)	8
4 (0)	0(2)	6(2)		10
4	6	8	6	

To test the initial solution for optimality (short-cut way). Compute the following table. Once the process is understood carefully, it will be easy to compute all the necessary information in one table only, instead of computing four tables for $[(u_i, v_j), \ (c_{ij}). \ (u_i + v_j), c_{ij} - (u_i + v_j)]$.

					u_i
	1 (1) 0+0	6 • (2)	1 (3) 0+2	4 (4) 0+0	0
	4 (4) 0+0	1 (3) 0+2	2 • (2)	6 • (0)	0
	4 • (0)	0 • (2)	6 • (2)	1 (1) 0+0	0
v_j	0	2	2	0	

Each cell of above table is read as follows :

1. The numbers written at the centre of the (*occupied*) cells are the allocations x_{ij} in the current solution. Occupied cells are also called ***basic-cells*** indicated by '•' at the centre of the basic cells.
2. **Down-left corner.** The numbers in the parenthesis written at the down-left corners of the cells are the unit transportation costs (c_{ij}).
3. **Down-right corner.** The numbers written at the down-right corners of (*empty*) cells are the *sum* of numbers u_i and v_j [*i.e.*, $u_i + v_j$]. The numbers u_i and v_j are calculated so that $c_{ij} = u_i + v_j$ for each *occupied cell.*
4. **Upper-right corner.** The numbers written at the upper-right corners of each *empty* cell are the net evaluations $d_{ij} = (c_{ij}) - (u_i + v_j)$ which are simply obtained by subtracting the number $u_i + v_j$ (at the down-right corner) from the corresponding unit cost (c_{ij}) written at the down-left corner. The cell evaluations are computed as follows :

$d_{11} = c_{11} - (u_1 + v_1) = (1) - (0+0) = 1$ $\quad$ $d_{14} = c_{14} - (u_1 + v_4) = (4) - (0+0) = 4$

$d_{21} = c_{21} - (u_2 + v_1) = (4) - (0+0) = 4$ $\quad$ $d_{22} = c_{22} - (u_2 + v_2) = (3) - (0+2) = 1$

$d_{13} = c_{13} - (u_1 + v_3) = (3) - (0+2) = 1$ $\quad$ $d_{34} = c_{34} - (u_3 + v_4) = (1) - (0+0) = 1$

These calculations can be done within the table orally.

From these calculations, it is observed that all cell evaluations (d_{ij}) are positive.

Hence the solution under test is optimal.

Example 8. *Given the following data.*

The cost of shipment from third source to the third destination is not known. How many units should be transported from sources to the destinations so that the total cost of transporting all the units to their destinations is a minimum.

		Destinations 1	2	3	Capacity
Sources	1	2	2	3	10
	2	4	1	2	15
	3	1	3	×	40
Demand		20	15	30	

Solution. Since the cost c_{33} is unknown, assign a large cost, say M, to this cell. Now using *Vogel's method* an initial basic feasible solution is obtained.

Table 11·54
Initial B.F.S.

(2)	(2)	**10**(3)
(4)	(1)	**15**(2)
20(1)	**15**(3)	5(M)

Table 11·55
$(u_i + v_j)$ Matrix

				u_i
	4 − M	6 − M	• (3)	3 − M
	3 − M	5 − M	• (2)	2 − M
	• (1)	• (3)	• (M)	0
v_j	1	3	M	

Table 11·56
Matrix $[c_{ij} - (u_i + v_j)]$

				u_i
	2 − (4 − M) + ive	2 − (6 − M) + ive	•	3 − M
	4 − (3−M) + ive	1 − (5 − M) + ive	•	2 − M
	•	•	•	0
v_j	1	3	M	

Since all the net evaluations for empty cells are positive, the current solution is optimum. But, it is to be noticed here that cell (3, 3) also appears in the solution for which the cost for shipment is not known. Hence, there exists a ***pseudo optimum basic feasible solution*** : $x_{13} = 10$, $x_{23} = 15$, $x_{31} = 20$, $x_{23} = 15$ and $x_{33} = 5$.

Example 9. *Is* $x_{13} = 50, x_{14} = 20, x_{21} = 55, x_{31} = 30$, $x_{32} = 35$, $x_{34} = 25$ *an optimum solution of the following transportation problem ?*

	To				Available units
From	6	1	9	3	70
	11	5	2	8	55
	10	12	4	7	90
Required units	85	35	50	45	

*If not, modify it to obtain a **between** feasible solution.* **[Meerut (Maths) 91]**

Solution. The initial feasible solution is given as in the following table with cost of Rs. 1460.

		50(9)	20(3)
55(11)			
30(10)	35(12)		25(7)

Starting Table. *Vacate the cell (3, 4) and occupy (2, 3).*
[θ = min (50 , 55 , 25) = 25]

Matrix for u_i and v_j

					u_i
			• (9)	• (3)	−4
	• (11)				1
	• (10)	• (12)		• (7)	0
v_j	10	12	13	7	

First Iteration Table. *Vacate the cell (1, 3) and occupy (1, 2).* Here θ = min [35 , 30 , 25] = 25.

					u_i
	0 (6) 6	−7 (1) 8	50 − θ (9)	20 + θ (3)	−4
	55 − θ (11)	−8 (5) 13	−12 +θ√ (2) 14	0 (8) 8	1
	30 + θ (10)	35 (12)	−9 (4) 13	25 − θ (7)	0
v_j	10	12	13	7	

					u_i
	−12 (6) 18	+θ −19 √ (1) 20	25 − θ (9)	45 (3)	0
	30 − θ (11)	−8 (5) 13	25 + θ (2)	12 (8) −4	−7
	55 + θ (10)	35 − θ (12)	3 (4) 1	3 12 (7) −5	−8
v_j	18	20	9	3	

Second Iteration Table. *Vacate cell (2, 1) and occupy (2, 2).* Here $\theta = \min(5, 10) = 5$.

					u_i
	7 (6) −1	25 (1)	19 (9) −10	45 (3)	0
	5 − θ (11)	θ √ −8 (5) 13	50 (2) 14	−7 (8) 15	12
	80 + θ (10)	10 − θ (12)	3 (4) 1	−7 (7) 14	11
v_j	−1	1	−10	3	

Third Iteration Table. *Vacate the cell (3, 2) and occupy (3, 4).* Here $\theta = \min[45, 5] = 5$.

					u_i
	7 (6) −1	25 + θ (1)	11 (9) −2	45 − θ (3)	0
	8 (11) 3	5 (5)	50 (2)	1 (8) 7	4
	85 (10)	5 − θ (12)	−5 (4) 9	−7 √ +θ (7) 14	11
v_j	−1	1	−2	3	

Since all the net evaluations are non-negative, an optimum basic feasible solution is obtained as :
$x_{12} = 30$, $x_{14} = 40$, $x_{22} = 5$, $x_{23} = 50$,
$x_{31} = 85$, and $x_{34} = 5$.
$z = 30 \times 1 + 40 \times 3 + 5 \times 5 + 50 \times 2 + 85 \times 10 + 5 \times 7$
= Rs. 1160.

Optimal Iteration Table

					u_i
	0 (6) 6	30 (1)	11 (9) −2	40 (3)	0
	1 (11) 10	5 (5)	50 (2)	1 (8) 7	4
	85 (10)	7 (12) 5	2 (4) 2	5 (7)	4
v_j	6	1	−2	3	

Example 10. *Solve the transportation problem where all entries are unit costs.*

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	73	40	9	79	20	8
O_2	62	93	96	8	13	7
O_3	96	65	80	50	65	9
O_4	57	58	29	12	87	3
O_5	56	23	87	18	12	5
b_j	6	8	10	4	4	

Solution. Using *'Lowest Cost Entry Method,* the initial basic feasible solution having transportation cost Rs. 1123 is obtained as below.

		8(9)			8
3(62)			4(8)		7
2(96)	7(65)				9
1(57)		2(29)			3
	1(23)			4(12)	5
6	8	10	4	4	

Starting Iteration Table

					u_i
+ (73) 37	+ (40) 6	8 • (9)	+ (79) − 17	+ (20) − 5	− 25
3 − θ (62)	+ (93) 31	+ (96) 34	4 • (8)	− 7 + θ √ (13) 20	0
2 + θ (96)	7 − θ (65)	+ (80) 68	+ (50) 42	+ (65) 54	34
1 • (57)	+ (58) 26	2 • (29)	+ (12) 3	+ (87) 15	− 5
+ (56) 54	1 + θ (23)	+ (87) 26	+ (18) 0	4 − θ (12)	− 8
v_j: 62	31	34	8	20	

Here θ = min [3, 4, 7] = 3.

First Iteration Table. *Vacate the cell (2, 1) and occupy (2, 5).*

					u_i
+ (73) 37	+ (40) 6	8 • (9)	+ (79) − 10	+ (20) − 5	− 18
+ (62) 55	+ (93) 24	+ (96) 27	4 • (8)	3 • (13)	0
5 • (96)	4 • (65)	+ (80) 68	+ (50) 49	+ (65) 54	41
1 • (57)	+ (58) 26	2 • (29)	+ (12) 10	+ (87) 15	2
+ (56) 54	4 • (23)	+ (87) 26	+ (18) 7	1 • (12)	− 1
v_j: 55	24	27	8	13	

Since all the net evaluations at the upper-right corners of all empty cells are non-negative, the solution under test is optimal. Hence optimum allocation is given by, $x_{13} = 8$, $x_{24} = 4$, $x_{25} = 3$, $x_{31} = 5$, $x_{32} = 4$; $x_{41} = 1$, $x_{43} = 2$, $x_{52} = 4$ and $x_{55} = 1$, and minimum transportation cost

$$z = 8 \times 9 + 4 \times 8 + 3 \times 13 + 5 \times 96 + 4 \times 65 + 1 \times 57 + 2 \times 29 + 4 \times 23 + 1 \times 12 = \text{Rs. } 1102 .$$

Example 11. *The following table shows all the necessary information on the available supply to each warehouse, the requirement of each market and the unit transportation cost from each warehouse to each market.*

		Market				
		I	II	III	IV	Supply
Warehouse	A	5	2	4	3	22
	B	4	8	1	6	15
	C	4	6	7	5	8
Requirement		7	12	17	9	

The shipping clerk has worked-out the following schedule from experience : 12 units from A to II, 1 unit from A to III, 9 units from A to IV, 15 units from B to III, 7 units from C to I, and 1 unit from C to III.

(a) *Check and see if the clerk has the optimal schedule,*

(b) *Find the optimal schedule and minimum total shipping cost.*

(c) *If the clerk is approached by a carrier of route C to II who offers to reduce his rate in the hope of getting some business, by how much must the rate be reduced before the clerk should consider giving him an order.* **[VTU (BE Mech.) 2003]**

Solution. The basic feasible solution is given as follows :

	12 (2)	1 (4)	9 (3)	22
		15 (1)		15
7 (4)		1 (7)		8
7	12	17	9	

(a) Starting Iteration Table.

				u_i
+ (5) 1	12 • (2)	1 + θ (4)	9 − θ (3)	0
6 (4) −2	+ (8) −1	• 15 (1)	6 (6) 0	−3
7 • (4)	+ (6) 5	1 − θ (7)	−1 + θ √ (5) 6	3
v_j 1	2	4	3	

(b) Optimal Table

				u_i
+ (5) 2	12 • (2)	2 • (4)	8 • (3)	0
+ (4) −1	+ (8) −1	15 • (1)	6 (6) 0	−3
7 • (4)	+ (6) 4	+ (7) 6	1 • (5)	2
v_j 2	2	4	3	

Here $\theta = \min [9, 1] = 1$.

Since all the net evaluations (at the upper-right corners) of empty cells are non-negative, the current solution is optimal.

Hence the optimal schedule is : $x_{12} = 12$, $x_{13} = 2$, $x_{14} = 8$, $x_{23} = 15$, $x_{31} = 7$, $x_{34} = 1$.

Minimum total shipping cost will be $z = 12 \times 2 + 2 \times 4 + 8 \times 3 + 15 \times 1 + 7 \times 4 + 1 \times 5 = \text{Rs. } 104$.

(c) If the clerk decides to transport at the most 8 units from *C* to *II* (instead of 7 to *I* and 1 to *IV*), then *II* may reduce his cost from $c_{32} = 6$ to at least 4 in order to have the improved cost. So according to the given proposal, the total minimum transportation cost will become Rs. 103.

Example 12. *The following table gives the cost for transporting material from supply points A, B, C and D to demand points E, F, G, H, and J.*

		To				
		E	F	G	H	J
	A	8	10	12	17	15
	B	15	13	18	11	9
From	C	14	20	6	10	13
	D	13	19	7	5	12

The present allocation is as follows :

A to E 90 ; A to F 10 ; B to F 150 ; C to F 10 ; C to G 50 ; C to J 120 ; D to H 210 ; D to J 70.

(a) *Check if this allocation is optimum. If not, find an optimum schedule.*

(b) *If in the above problem, the transportation cost from A to G is reduced to 10, what will be the new optimum schedule ?* **[IAS (Main) 91]**

Solution. The given allocation has the transportation cost = Rs. 6930. Now test the given solution for optimality as follows :

Starting Iteration Table

					u_i
(8) 90 − θ	(10) 10 + θ	(12) + −4	(17) + −3	(15) + 3	10
(15) + 11	(13) 150	(18) + −1	11 + 0	9 + 6	13
(14) −4 θ √ 18	(20) 10 − θ	(6) 50	(10) + 7	(13) 120	20
(13) −4 17	(19) 0 19	(7) + 5	(6) 210	(12) 70	19
v_j: −2	0	−14	−13	−7	

Here θ = min [90 , 10] = 10.

First Iteration Table. *Vacate the cell (3, 2) and occupy (3, 1).*

					u_i
(8) 80 − θ	(10) 20 + θ	(12) + 0	(17) + 1	15 + 7	−6
(15) + 11	(13) 150 − θ	(18) + 3	11 + 4	(9) −1 + θ √ 10	−3
(14) 10 + θ	(20) + 16	(6) 50	(10) 7	(13) 120 − θ	0
(13) 0 13	(19) + 15	(7) + 5	(6) 210 5	(12) 70	−1
v_j: 14	16	6	7	13	

Here θ = min [120, 80, 150] = 80,

Second Iteration (Optimal) Table

					u_i
(8) + 7	(10) 100 ●	(12) + −1	(17) + 0	(15) + 6	−3
(15) + 10	(13) 70 ●	(18) + 2	(11) + 3	(9) 80 ●	0
(14) 90 ●	(20) + 17	(6) 50 ●	(10) + 7	(13) 40 ●	4
(13) 0 13	(19) + 16	(7) + 5	(6) 210 ●	(12) 70 ●	3
v_j 10	13	2	3	9	

Since all the net-evaluations are non-negative, the following optimum solution is obtained :

$$x_{12} = 100, x_{22} = 70, x_{25} = 80, x_{31} = 90, x_{33} = 50, x_{35} = 40, x_{44} = 210, x_{45} = 70,$$

and minimum cost = Rs. 6810.

Example 13. *Hindustan Construction Company needs 3, 3, 4 and 5 million cubic feet of fill at four earthen dam-sites in Punjab. It can transfer the fill from three mounds A, B and C where 2, 6 and 7 million cubic feet of fill is available respectively. Costs of transporting one million cubic feet of fill from mounds to the four sites in lakhs are given in the table.*

(a) Solve the problem using transportation algorithm for minimum cost :

(b) *Formulate the problem as L.P.P.*

To

		I	II	*III*	*IV*	a_i
	A	15	10	17	18	2
From	*B*	16	13	12	13	6
	C	12	17	20	11	7
	b_j	3	3	4	5	

Solution. Using "*Vogel's Method*" the initial B.F.S with cost Rs. 190 is obtained as follows :

		2(17)	
	3(13)	2(12)	1(13)
3(12)			4(11)

Starting Iteration Table

				u_i
(15) −4 19	(10) +θ √ −8 18	(17) 2 − θ	(18) 0 18	5
(16) + 14	(13) 3 − θ	(12) 2 + θ	(13) 1 ●	0
(12) 3 ●	(17) + 11	(20) + 10	(11) 4 ●	−2
v_j 14	13	12	13	

Hence θ = min [3, 2] = 2.

Since all the net-evaluations are not non-negative, the current solution is not optimal.

First Iteration Table. *Vacate the cell (1, 3) and occupy (1, 2).*

					u_i
	(15) + 11	(10) 2 •	(17) + 9	(18) + 10	−3
	(16) + 14	(13) 1 •	(12) 4 •	(13) 1 •	0
	(12) 3 •	(17) + 11	(20) + 10	(11) 4 •	−2
v_j	14	13	12	13	

Since all the net-evaluations are non-negative, the optimum solution is

$$x_{12} = 2, x_{22} = 1, x_{23} = 4, x_{24} = 1, x_{31} = 3, x_{34} = 4,$$

and minimum cost z = Rs. 174.

L.P. Formulation. Let x_{ij} be the amount of fill transferred from mounds to dam-sites. Then formulation of the problem becomes :

$$\text{Min } z = (15x_{11} + 10x_{12} + 17x_{13} + 18x_{14}) + (16x_{21} + 13x_{22} + 12x_{23} + 13x_{24}) + (12x_{31} + 17x_{32} + 20x_{33} + 11x_{34})$$

subject to the constraints :

$$x_{11} + x_{12} + x_{13} + x_{14} = 2 \qquad x_{12} + x_{22} + x_{32} = 3$$
$$x_{21} + x_{22} + x_{23} + x_{24} = 6 \qquad x_{13} + x_{23} + x_{33} = 4$$
$$x_{31} + x_{32} + x_{33} + x_{34} = 7 \qquad x_{14} + x_{24} + x_{34} = 5$$
$$x_{11} + x_{21} + x_{31} = 3 \qquad \text{and all } x_{ij} \geq 0 \; (i = 1, 2, 3; j = 1, 2, 3, 4)$$

EXAMINATION PROBLEMS

1. Find the optimum solution to the following transportation problem for which the cost, origin-availabilities, and destination-requirements are as given below :

(i) To

	A	B	C	D	E	a_i
I	3	4	6	8	8	20
II	2	10	1	5	30	30
From III	7	11	20	40	15	15
IV	2	1	9	14	18	13
$b_j \rightarrow$	40	6	8	18	6	

[Meerut M.Sc (Maths.) Jan. 98 BP, 94]

[Hint. First find the starting solution by "Vogel's method" and then show that this is an optimal solution].
[Ans. $x_{11} = 14, x_{15} = 6, x_{21} = 4, x_{23} = 8, x_{24} = 18, x_{31} = 15, x_{41} = 7, x_{42} = 6$; min cost = Rs. 321.]

(ii)

	D_1	D_2	D_3	D_4	D_5	D_6	a_i	
O_1	1	2	1	4	5	2	30	
O_2	3	3	2	1	4	3	50	
O_3	4	2	5	9	6	2	75	
O_4	3	1	7	3	4	6	20	
b_j	20	40	30	10	50	25	175	Total

[Hint. Find the initial solution by 'Lowest Cost Entry Method'. Although, "Vogel's method" will also yield the same initial solution, but will be longer one. One improvement is only required to reach the optimal solution.]
[Ans. $x_{11} = 20, x_{13} = 10, x_{23} = 20, x_{24} = 10, x_{25} = 20, x_{32} = 40, x_{35} = 10, x_{36} = 25, x_{45} = 20$, min cost = Rs. 430.]

(iii)

Store

		A	B	C	a_i
Factory	I	10	9	8	8
	II	10	7	10	7
	III	11	9	7	9
	IV	12	14	10	4
	$b_j \to$	10	10	8	

[Hint. Find initial BFS by 'Lowest Cost Entry Method' and prove it to be optimal]

[Ans. $x_{11} = 6$,. $x_{12} = 2$, $x_{22} = 7$, $x_{32} = 1$, $x_{33} = 8$, $x_{41} = 4$, min. cost = Rs. 240]

(iv)

	D_1	D_2	D_3	D_4	Supply
O_1	1	2	−2	3	70
O_2	2	4	0	1	38
O_3	1	2	−2	5	32
Demand	40	28	30	32	

[Hint. Use 'VAM' to find initial BFS with cost Rs. 86.]

[Ans. $x_{11} = 40$, $x_{12} = 26$, $x_{14} = 4$, $x_{24} = 38$, $x_{32} = 2$ and $x_{33} = 30$; min cost = Rs. 86].

(v)

	D_1	D_2	D_3	D_4	a_i
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
$b_j \to$	16	18	31	25	

[Hint. Find initial BFS by 'VAM' and prove it to be optimal.

[Ans. $x_{12} = 18$, $x_{13} = 1$, $x_{21} = 12$, $x_{24} = 25$, $x_{31} = 4$, $x_{33} = 30$, min. cost = Rs. 355]

(vi)

To

		1	2	3	4	5	Supply
	1	2	3	5	7	5	17
From	2	4	1	2	1	6	13
	3	2	8	6	1	3	16
	4	5	3	7	2	4	20

[Hint. By VAM we get degeneracy. Find initial solution by *Lowest Cost Entry Method*, then prove optimum.]

(vii)

To

		1	2	3	a_i
	I	2	7	4	5
	II	3	3	7	8
From	III	5	4	1	7
	IV	1	6	2	14
	$b_j \to$	7	9	18	

[Meerut (B. Sc.) 90]

[Hint. Find initial solution by VAM and and revise once optimum.]

[Ans. $x_{11} = 5$, $x_{22} = 8$, $x_{32} = 1$, $x_{33} = 6$, $x_{41} = 2$, $x_{43} = 12$, min. cost = Rs. 70]

(viii)

To

		1	2	3	4	Supply
	1	5	3	6	4	30
From	2	3	4	7	8	15
	3	9	6	5	8	15
Demand		10	25	18	7	60 (Total)

[Hint. Find initial solution by VAM and it will be proved optimum. to get. Alternative solutions also exist.]

[Ans. (i) $x_{12} = 20$, $x_{13} = 3$, $x_{14} = 7$, $x_{21} = 10$, $x_{22} = 5$, $x_{33} = 15$, min. cost = Rs. 231 (ii) $x_{12} = 23$, $x_{14} = 7$, $x_{21} = 10$, $x_{22} = 2$, $x_{23} = 3$, $x_{33} = 15$

(ix)

	D_1	D_2	D_3	D_4	Supply
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
Demand	22	35	25	41	123 (Total)

[Hint. Find initial solution by VAM, it will be proved optimum]

[Ans. $x_{14} = 30$, $x_{21} = 5$, $x_{22} = 35$, $x_{31} = 17$, $x_{33} = 25$, $x_{34} = 11$, cost = Rs. 1921

(x)

	D_1	D_2	D_3	D_4	Supply
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
Demand	16	18	31	25	

[Mudurai BSc (Comp. Sc.) 92]

[Ans. $x_{12} = 18$, $x_{13} = 1$, $x_{21} = 12$, $x_{24} = 25$, $x_{31} = 4$, $x_{33} = 33$, min cost = 355]

(xi)

	D_1	D_2	D_3	D_4	s_i
O_1	3	8	9	16	8
O_2	6	11	14	6	9
O_3	5	13	10	12	13
d_j	6	7	7	10	

[Ans. $x_{11} = 4$, $x_{12} = 4$, $x_{24} = 9$, $x_{31} = 2$, $x_{32} = 3$, $x_{33} = 7$, $x_{34} = 1$, min cost = 256]

(xii)

	I	II	III	IV	Supply
A	6	3	5	4	22
B	5	9	2	7	15
C	5	7	8	6	8
Requirement	7	12	17	9	

[Ans. $x_{12} = 12$, $x_{13} = 2$, $x_{14} = 8$, $x_{23} = 15$, $x_{31} = 7$, $x_{34} = 1$, min. cost = 149]

2. Goods have to be transported from factories F_1, F_2, F_3, to ware-hourse W_1, W_2, W_3 and W_4. The transportation cost per unit, capacities and requirement of the warehouse are given in the following table.

	W_1	W_2	W_3	W_4	Capacity
F_1	95	105	80	15	12
F_2	115	180	40	30	7
F_3	195	180	95	70	5
Requirement	5	4	4	11	

Find the distribution with minimum cost.

3. Solve the following cost-minimizing transportation problem.

	D_1	D_2	D_3	D_4	D_5	D_6	Available
O_1	2	1	3	3	2	5	50
O_2	3	2	2	4	3	4	40
O_3	3	5	4	2	4	1	60
O_4	4	2	2	1	2	2	30
Required	30	50	20	40	30	10	180

[JNTU (B. Tech.) 2003; Delhi B.Sc. (Maths.) 91]

[Ans. $x_{12} = 50$, $x_{23} = 20$, $x_{31} = 30$, $x_{34} = 20$, $x_{36} = 10$, $x_{44} = 20$, $x_{45} = 10$, min cost = 330]

4. Consider four *bases* of operations B_i and three *targets* T_j. The tops of bombs per aircraft from any base that can be delivered to any target are given in the table below.

	T_1	T_2	T_3
B_1	8	6	5
B_2	6	6	6
B_3	10	8	4
B_4	8	6	4

The daily sortie *capacity* of each of the four bases is 150 sorties per day. The daily *requirement* in sorties over each individual target is 200. Find the allocation of sorties from each base to each target which maximizes the total tonnage over all three targets explaining each step.

5. There are three sources or origins which store a given product. These sources supply these products to four dealers. The capacities of the sources and the demands are as given below :

$S_1 = 150$, $S_2 = 40$, $S_3 = 80$ $D_1 = 90$, $D_2 = 70$, $D_3 = 50$, $D_4 = 60$.

The cost of transporting the product from various sources to various dealers is shown in the table below.

	D_1	D_2	D_3	D_4
S_1	27	23	31	69
S_2	10	45	40	32
S_3	30	54	35	57

Find out the optimum solution for transporting the products at a minimum cost.

[Hint. Find initial BFS by VAM and prove it to be optimal.]

[Ans. $x_{11} = 30$, $x_{12} = 70$, $x_{13} = 50$, $x_{24} = 40$, $x_{31} = 60$, $x_{34} = 20$, min cost = 8190]

6. A firm manufacturing a single product has plants I, II, and III. The three plants have produced 60, 35 and 40 units respectively during this month. The firm had made a commitment to sell 2 units to customer A, 45 units to customer B, 20 units to customer C, 18 units to customer D, and 30 units to customer E. Find the minimum possible transportation cost of shipping the manufactured product to five customers. The net per unit cost of transporting from the three plants to five customers is given in the table. **[JNTU (Mech. & Prod.) 2004]**

		Customer				
		A	*B*	C	*D*	*E*
	I	4	1	3	4	4
Plant	*II*	2	3	2	2	3
	III	3	5	2	4	4

[Hint. Find initial BFS by VAM and prove it to be optimal.]

[Ans. $x_{12} = 45$, $x_{15} = 15$, $x_{21} = 17$, $x_{24} = 18$, $x_{31} = 5$, $x_{33} = 20$, $x_{35} = 15$, min. cost = Rs. 290].

7. The figures in the body of the table below are proportional to the cost of transportation of the tonne of food grain from the port given by the row heading to the destination given by column heading.

Ports	Delhi	Hyderabad	Mysore	Nagpur	stock (in thousand tonnes)
Bombay	9	5	8	5	225
Calcutta	9	10	13	7	75
Madras	14	5	3	7	100
Requirements (thousand tonnes)	125	80	95	100	400

Plan a transportation scheme satisfying the requirements of each destination and at the same time minimizing the total transportation cost.

[**Hint.** Find the initial BFS by VAM and prove it to be optimal. Alternative solutions exists.]

[**Ans.** $x_{11} = 50$, $x_{12} = 75$, $x_{14} = 100$, $x_{21} = 75$, $x_{32} = 5$, $x_{33} = 95$, min. cost = 2310]

8. An oil corporation has got three refineries P, Q, and R and it has to send petrol to four different depots A, B, C and D. The cost of shipping 1 gal. of petrol and the available petrol at the refineries are given in the table. The requirement of the depots and the available petrol at the refineries are also given. Find the minimum cost of shipping after obtaining an initial solution by VAM.

		Depot				
		A	B	*C*	*D*	Available
	P	10	12	15	8	130
Refinery	*Q*	11	11	9	10	150
	R	20	9	7	18	170
Required		90	100	140	120	

[**Hint.** Find initial BFS by VAM and prove it to be optimal.]

[**Ans.** $x_{11} = 90$, $x_{14} = 40$, $x_{23} = 70$, $x_{24} = 80$, $x_{32} = 100$, $x_{33} = 70$; min. cost = Rs. 3460]

9. King Mahendra of the Pallavedynasty Red sculptors stationed at Kanchipuram, Mallapuram and Tiruchirapalli. There were 15 sculptors at Kanchipuram, 30 at Mallapuram and 5 at Tiruchirapalli. Rock-cut shrines had to be excavated and 12 sculptors were required at Mandagapattu, 20 at Pallavapuram, 8 at seevamangalam and 10 at Mahendravadi, give a solution which minimizes the cost of transport of the sculptors assuming that the cost of transport per mile is the same for all the routes. The distance in miles between the different places is given in the following table.

From	To			
	Mandagapatta	Pallavapuram	Seevaman-galam	Mahandravadi
Kanchipuram	60	40	50	20
Mallapuram	70	30	65	55
Tiruchirapalli	100	200	120	210

[**Ans.** $x_{13} = 5$, $x_{14} = 10$, $x_{21} = 7$, $x_{22} = 20$, $x_{23} = 3$, $x_{31} = 5$; min distance = 2235 miles.]

10. A company has factories at F_1, F_2 and F_3 which supply warehouses at W_1, W_2 and W_3. Weekly factory capacities are 200, 160 and 90 units respectivley. Weekly warehouses requirements are 180, 120 and 150 units respectivley. Unit shipping costs (in rupees) are as follows :

		Warehouse			
		W_1	W_2	W_3	Supply
	F_1	16	20	12	200
Factory	F_2	14	8	18	160
	F_3	26	24	16	90
	Demand	180	120	150	350

Determine the optimum distribution for this company to minimize shipping costs. **[Kerala (M. Com.) 97)**

[**Ans.** F_1 : $x_{12} = 140$, $x_{12} = 0$, $x_{13} = 60$, F_2 : $x_{21} = 40$, $x_{22} = 120$, $x_{23} = 0$

F_3 : $x_{31} = 0$, $x_{32} = 0$, $x_{33} = 90$ Total transportation cost = Rs. 5,920]

11. A wholesale company has three warehouses from which supplies are drawn for four retail customers. The company deals in a single product, the supplies of which at each warehouse are :

Warehouse no.	Supply (units)	Customer no.	Demand (units)
1	20	1	15
2	28	2	19
3	17	3	13
		4	18

Conveniently, total supply at the warehouses is equal to total demand from the customers. The following table gives the transportation costs per unit shipment from each warehouse to each customer.

Warehouse	Customer			
	1	2	3	4
1	3	6	8	5
2	6	1	2	5
3	7	8	3	9

Determine what supplies to despatch from each of the warehouses to each customer so as to minimize overall transportation cost. **[AIMA (Dip. in Management) Dec. 1996]**

[**Ans.** $x_{11} = 15$, $x_{14} = 5$, $x_{22} = 19$, $x_{24} = 9$, $x_{33} = 13$, $x_{34} = 4$. Total transportation cost = Rs. 209.]

12. A manufacturer has distribution centres at *X, Y,* and *Z*. These centres have availability 40, 20 and 40 units of his product. His retail outlets at *A, B, C, D* and *E* requires 25, 10, 20, 30 and 15 units respectively. The transport cost (in rupees) per unit between each centre outlet is given below :

Distribution centre	Retail outlets				
	A	B	C	D	E
X	55	30	40	50	50
Y	35	30	100	45	60
Z	40	60	95	35	30

Determine the optimal distribution to minimize the cost of transportation. **[Poone (M.B.A.) 97]**

[**Ans.** $x_{11} = 5$, $x_{12} = 10$, $x_{13} = 20$, $x_{14} = 5$, $x_{21} = 20$, $x_{44} = 29$, $x_{45} = 15$, Total transportation cost = Rs. 3650]

13. *ABC* limited has three production shops supplying a product ot five warehouses. The cost of production varies from shop to shop and cost of transportation from one shop to a warehouse also varies. Each shop has a specific production capacity and each warehouse has certain amount of requirement. The cost of transportation are as given below :

Shop	Warehouse					Capacity
	I	II	III	IV	V	
A	6	4	4	7	5	100
B	5	6	7	4	8	125
C	3	4	6	3	4	175
Requirement	60	80	85	105	70	400

The cost of manufacture of the product at different production shops is :

Shop	Variable cost	Fixed cost
A	14	7,000
B	16	4,000
C	15	5,000

Find the optimum quantity to be supplied from each shop to different warehouses at minimum total cost.

[Nagpur (M.B.A.) Nov. 98]

[**Ans.** $x_{12} = 15$, $x_{13} = 85$, $x_{22} = 20$, $x_{24} = 105$, $x_{31} = 60$, $x_{32} = 45$, $x_{35} = 70$ Min. total transportation cost = 7605]

14. A company has three factories at Amethi, Baghpat and Gwalior; and four distribution centres at Allahabad, Bombay, Calculta and Delhi. With identical cost of production at the three factories the only variable cost involved is transportation cost. The production at the three factories is 5,000 tonnes; 6,000 tonnes; 2,500 tonnes respectively. The demand at four distribution centres is 6,000 tonnes; 4,000 tonnes; 2,000 tonnes and 1,500 tonnes respectively. The transportation costs per tonne from different factories to different centres are given below :

Factory	Distribution Centre			
Amethi	3	2	7	6
Baghpat	7	5	2	3
Gvalior	2	5	4	5

Suggest the optimum transportation schedule and find the minimum cost of transportation.

[Gujarat (MBA) 96; Delhi (M.Com.) 95]

[**Ans.** $x_{11} = 3500$, $x_{12} = 1500$, $x_{22} = 2500$, $x_{23} = 2000$, $x_{24} = 1500$ and $x_{31} = 2500$, Min. transportation cost = Rs. 39,500.]

15. A company manufacturing air-coolers has two plants located at Bombay and Calcutta with a weekly capacity of 200 units and 100 units, respectively. The company supplies air coolers to its 4 show rooms situated at Ranchi, Delhi, Lucknow and Kanpur which have a demand of 75,100, 100 and 30 units, respectively. The cost of transportation per unit (in Rs.) is shown in the following table :

	Ranchi	Delhi	Lucknow	Kanpur
Bombay	90	90	100	100
Calcutta	50	70	130	85

Plant the production programme so as to minimize the total cost of transportation. **[Rajasthan (M.Com.) 97]**
[**Ans.** $x_{12} = 75$, $x_{13} = 95$, $x_{14} = 30$, $x_{21} = 75$, $x_{22} = 25$, $x_{31} = 30$, $x_{32} = 10$, $x_{33} = 0$.
Min. transportation cost = Rs. 24750. Alternative optimum solution also exists.]

16. The *ABC* Tool Company has a sales force of 25 men who work out from three regional offices. The company produces four basic product lines of hand tools. Mr. Jain, the sales manager feels that 6 salesmen are needed to distribute product line 1, 10 salesmen to distribute product line 2, 4 salesmen for product line 3 and 5 salesmen for product line 4. The cost (in Rs.) per day of assigning salesmen from each of the offices for selling each of the product lines are as follows :

	Product lines			
	1	2	3	4
Regional office A	Rs. 20	Rs. 21	Rs. 16	Rs. 18
Regional office B	17	28	14	16
Regional office C	29	23	19	20

At the present time, 10 salesmen are allocated to office A, 9 salesmen to office B, and 7 salesmen to office C. How many salesmen should be assigned from each office to sell each product line in order to minimize costs ? Identify alternate optimum solutions, if any. **[Delhi (MBA,) April 98)**
[**Ans.** $x_{12} = 4$, $x_{13} = 1$, $x_{14} = 5$, $x_{21} = 6$, $x_{23} = 3$, $x_{32} = 6$, $x_{35} = 1$, Min. transportation cost = Rs. 472.]

17. The Purchase Manager, Mr. Shah, of the State Road Transport Corporation must decide on the amounts of fuel to buy from three possible vendors. The corporation refuels its buses regularly at the four depots within the area of its operations.
The three oil companies have said that they can furnish up to the following amounts of fuel during the coming month : 275,000 litres by oil company 1 : 50,000 litres by oil company 2; and 660,000 litres by oil company 3. The required amount of the fuel is 110,000 litres by depot 1; 20,000 litres at depot 2; 330,000 litres at depto 3; and 440,000 litres at depot 4.
When the transportation costs are added to the bid price per litre supplied, the combined cost per litre for fuel from each vendor servicing a specific depot is shown below :

	Company 1	Company 2	Company 3
Depot 1	5.00	4.75	4.25
Depot 2	5.00	5.50	6.75
Depot 3	4.50	6.00	5.00
Depot 4	5.50	6.00	4.50

Determine the optimum transportation schedule. **[Delhi (M.B.A.,) Nov. 97)**
[**Ans.** $x_{13} = 110$, $x_{21} = 55$, $x_{22} = 165$, $x_{31} = 220$, $x_{33} = 110$, $x_{43} = 440$, $x_{52} = 385$, Min. transportation cost = Rs. 51,70,000.]

11.11. DEGENERACY IN TRANSPORTATION PROBLEMS

The solution procedure for non-degenerate basic feasible solution with exactly $m + n - 1$ strictly positive allocations in *independent positions* has been discussed so far. However, sometimes it is not possible to get such initial feasible solution to start with. Thus degeneracy occurs in the transportation problem whenever a number of occupied cells is less than $m + n - 1$.

We recall that a basic feasible solution to an m-origin and n-destination transportation problem can have at most $m + n - 1$ number of positive (non- zero) basic variables. If this number is exactly $m + n - 1$, the BFS is said to *be non-degenerate;* and if less than $m + n - 1$ the basic solution degenerates. It follows that whenever the number of basic cells is less than $m + n - 1$, the transportation problem is a degenerate one.

Degeneracy in transportation problems can occur in two ways :

1. Basic feasible solutions may be degenerate from the initial stage onward.
2. They may become degenerate at any intermediate stage.

Q. 1. What is degeneracy problem in transportation ? What is its cause ? How it can be over come. **[JNTU (Mech. & Prod.) 2004]**

2. Explain briefly about unbalanced transportation problem and degenerate case in transportation problem. **[VTU (BE Mech.) 2002]**

11·11-1. Resolution of Degeneracy During the Initial Stage

To resolve degeneracy, allocate an extremely small amount of goods (close to zero) to *one* or *more* of the empty cells so that a number of occupied cells becomes $m + n - 1$. The cell containing this extremely small allocation is, of course, considered to be an occupied cell.

Rule : The extremely small quantity usually denoted by the Greek letter Δ (delta) [also sometimes by ε (epsilon)] is introduced in the *least cost* independent cell subject to the following assumptions. If necessary, two or more Δ's can be introduced in the least and second least cost independent cells.

1. $\Delta < x_{ij}$ for $x_{ij} > 0$. 2. $x_{ij} + \Delta = x_{ij} - \Delta$, $x_{ij} > 0$. 3. $\Delta + 0 = \Delta$.

*4. If there are more than one Δ's in the solution, $\Delta < \Delta'$, whenever Δ is *above* Δ'.
If Δ and Δ' are in the same row, $\Delta < \Delta'$ when *A is to the left of* Δ'.

Q. What do you mean by non-degenerate basic feasible solution of a transportation problem.

For example, $50 + \Delta = 50$, and $200 - \Delta = 200$. Of course, if Δ is subtracted from itself the result is assumed to be zero.

Following example will make the procedure clear.

Example 14. *A company has three plants A, B and C and three warehouses X, Y and Z. Number of units available at the plants is 60, 70 and 80, respectively. Demands at X, Y and Z are 50, 80 and 80, respectively. Unit costs of transportation are as follows :*

Table 11·56

	X	Y	Z
A	8	7	3
B	3	8	9
C	11	3	5

What would be your transportation plan ? Give minimum distribution cost. **[Garhwal 97]**

Solution. Step 1. First, write the given cost-requirement *Table 12·56* in the following manner :

Q. State the transportation problem in general terms and explain the problem of degeneracy. **[IAS (Main) 97]**

Table 11·57

	X	Y	Z	Available
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
Requirement	50	80	80	210 (Total)

Step 2. Using either the *"Lowest Cost Entry Method"* or *"Vogel's Approximation Method,* obtain the initial solution.

Step 3. Since the number of occupied cells in *Table 11·58* is 4, which does not equal to $m + n - 1$ (that is 5), the problem is degenerate at the very beginning, and so the attempt to assign u_i and v_j values to *Table 11·58* will not succeed. However, it is possible to resolve this degeneracy by addition of Δ to some suitable cell.

Table 11·58

				Available
			60(3)	60
	50(3)		20(9)	70
		80(3)		80
Requirement	50	80	80	

Generally, it is not possible to add Δ to any empty cell. But, Δ must be added to one of those empty cells which make possible the determination of a unique set of u_i and v_j. So choose such empty cell with careful judgement, for if the empty cell (1,1) is made an occupied cell (by addition of Δ) it will not be possible to assign u_i and v_j values. The allocations in cell (1, 1), (1, 3), (2, 1), and (2, 3) will become in non-independent (rather than independent) positions.

On the other hand, the addition of Δ to any of the cells (1, 2), (2, 2), (3, 1) and (3, 3) will enable us to resolve the degeneracy and allow to determine a unique set of u_i and v_j values. So proceed to resolve the degeneracy by allocating Δ to least cost independent empty cell (3, 3) as shown in *Table* 11·59.

Table 11·59

				Available
			60(3)	60
	50(3)		20(9)	70
		80(3)	Δ (5)	80 + Δ = 80
Requirement	50	80	80 + Δ = 80	

Now proceed to test this solution for optimality.

Step 4. Values of u_i and v_j will be obtained as shown in *Table* 11·60.

Once a unique set of u_i and v_j values has been determined, various steps of the transportation algorithm can be applied in a routine manner to obtain an optimal solution.

Table 11·60

				u_i
			• (3)	−2
	• (3)		• (9)	4
		• (3)	• (5)	0
v_j	−1	3	5	

Table 11·61
Matrix $[c_{ij}]$ for empty cells only

(8)	(7)	•
•	(8)	•
(11)	•	•

Table 11·62
Matrix $[u_i + v_j]$ for empty cells

−3	1	•
•	7	•
−1	•	•

Since all cell-evaluations in *Table 11·63* are positive, the solution under test is optimal. The real total cost of subsequent solutions did not happen to change in the example after Δ was introduced. In general, this will not be the case. In as much as the infinitesimal quantity Δ plays only an auxiliary role and has no significance, it is removed when the optimal solution is obtained. Hence, the final answer is given in *Table 11·64*

Table 11·63
Matrix $[c_{ij} - (u_i + v_j)]$

11	6	•
•	1	•
12	•	•

Table 11·64

		60(3)
50(3)		20(9)
	80(3)	

Thus minimum cost is : 180 + 150 + 180 + 240 = Rs. 750.

11·11-2. Resolution of Degeneracy During the Solution Stages

The transportation problem may also become degenerate during the solution stages. This happens when most favourable quantity is allocated to the empty cell having the largest negative cell-evaluation resulting in simultaneous vacation of two or more of currently occupied cells. To resolve degeneracy, allocate Δ to one or more of recently vacated cells so that the number of occupied cells is $m + n - 1$ in the new solution. This type of degeneracy can be explained below in ***Example 15.***

Example 15. *The cost-requirement table for the transportation problem is given as below :*

Table 11·65

	W_1	W_2	W_3	W_4	W_5	Available
F_1	4	3	1	2	6	40
F_2	5	2	3	4	5	30
F_3	3	5	6	3	2	20
F_4	2	4	4	5	3	10
Required	30	30	15	20	5	

Solution. By *'North-West-Corner Rule'*, the non-degenerate initial solution is obtained in Table 11·66 :

Table 11·66

	W_1	W_2	W_3	W_4	W_5	Available
F_1	30(4)	10(3)				40
F_2		20(2)	10(3)			30
F_3			5(6)	15(3)		20
F_4				5(5)	5(3)	10
Required	30	30	15	20	5	

Now, test this solution for optimality to get the following table in usual manner.

Table 11·67

Matrix for set of u_i and v_j

						u_i
	4	3				0
		2	3			−1
			6	3		2
				5	3	4
v_j	4	3	4	1	−1	

Table 11·68

Matrix $[c_{ij}]$ for empty cells

•	•	(1)	(2)	(6)
(5)	•	•	(4)	(5)
(3)	(5)	•	•	(2)
(2)	(4)	(4)	•	•

Table 11·69

Matrix $(u_i + v_j)$ for empty cells

•	•	4	1	−1
3	•	•	0	−2
6	5	•	•	1
8	7	8	•	•

Table 11·70

Matrix $[c_{ij} - (u_i + v_j)]$ for empty cells

•	•	−3	1	7
2	•	•	4	7
−3	0	•	•	1
−6√	−3	−4	•	•

Since, the largest negative cell evaluation is $d_{41} = -6$ (marked √), allocate as much as possible to this cell (4, 1). This necessitates shifting of 5 units to this cell (4, 1) as directed by the closed loop in *Table* 11·71.

Table 11·71

						Available
	30 − θ (4)	10 + θ (3)				40
		20 − θ (2)	10 + θ (3)			30
			5 − θ (6)	15 + θ (3)		20
	+ θ √			5 − θ (5)	5 •	10
Required	30	30	15	20	5	

Here maximum possible value of θ is obtained by usual rule :

min. [30 − θ , 20 − θ , 5 − θ , 5 − θ] = 0 *i.e.*, 5 − θ = 0 or θ = 5 units.

From ***Table* 11·71,** the revised solution becomes :

Table 11·72

						Available
	25(4)	15(3)				40
		15(2)	15(3)			30
			0*(6)	20(3)		20
	5(2)			0*(5)	5(3)	10
Required	30	30	15	20	5	

In this solution, the number of allocations becomes less than $m+n-1$ on account of simultaneous vacation of two cells [(3, 3), (4, 4), as indicated by *]. Hence this is a degenerate solution.

Now, this degeneracy may resolve by adding Δ to one of the recently vacated cells [(3, 3) or (4, 4)]. But in minimization problem, add Δ to recently vacated cell (4, 4) only, because it has the lowest shipping cost of Rs 5 per unit.

The rest of the procedure will be exactly the same as explained earlier. This way, the optimal solution can be obtained.

Table 11·73

						Available
	5(4)		15(1)	20(2)		40
		30(2)				30
	15(3)				5(2)	20
	10(2)					10
Required	30	30	15	20	5	

Note. Here the aim of finding an initial solution by '*North-West Corner Rule*' is only to show as to how degeneracy may arise during the solution stages. Otherwise, the optimal solution can be obtained immediately by using the initial solution by Vogel's method.

Example 16. *Solve the following transportation problem:*

		D_1	D_2	D_3	D_4	D_5	D_6	Available
	O_1	9	12	9	6	9	10	5
	O_2	7	3	7	7	5	5	6
From	O_3	6	5	9	12	3	11	2
	O_4	6	8	11	2	2	10	9
		4	4	6	2	4	2	22 (Total)

[Meerut (Maths.) 97P, (Stat.) 90]

Solution. Using 'VAM' the initial basic feasible solution having the transportation cost Rs. 112 is given below:

Initial BFS

		5(9)				5
	4(3)	Δ (7)			2(5)	$6+\Delta=6$
1(6)		1(9)				2
3(6)			2(2)	4(2)		9
4	4	$6+\Delta=6$	2	4	2	

Since the number of allocations (= 8) is less than $m+n-1$ (= 9), a very small quantity Δ may be introduced in the independent cell (2, 3), although least cost independent cell is (2, 5).

Optimum Table

							u_i
	+ (9) 6	+ (12) 5	5 (9)	+ (6) 2	+ (9) 2	+ (10) 7	2
	+ (7) 4	4 (3)	Δ (7)	+ (7) 0	+ (5) 0	2 (5)	0
	1 (6)	0 (5) 5	1 (9)	+ (12) 2	+ (3) 2	+ (11) 7	2
	3 (6)	+ (8) 5	+ (11) 9	2 (2)	4 (2)	+ (10) 7	2
v_j	4	5	7	0	0	5	

Since all the net evaluations are non-negative, the current solution is an optimum one. Hence the optimum solution is : $x_{13} = 5, x_{22} = 4, x_{26} = 2, x_{31} = 1, x_{41} = 3, x_{44} = 2, x_{45} = 4, x_{33} = 1.$

The optimum transportation cost is given by

$$z = 5(9) + 4(3) + \Delta\,(7) + 2(5) + 1(6) + 1(9) + 3(6) + 2(2) + 4(2) = \text{Rs. } 112 \quad (\text{since } \Delta \to 0).$$

Note. In above optimum table, Δ may also be introduced in least cost independent cell (2, 5).

Example 17. *Solve the following transportation problem (cell entries represent unit costs) :*

							Available
	5	3	7	3	8	5	3
	5	6	12	5	7	11	4
	2	1	3	4	8	2	2
	9	6	10	5	10	9	8
Required	3	3	6	2	1	2	17 (Total)

[Meerut (M.Com.) Jan. 98 (BP), (M.Sc.) 93 P]

Soluton. Using 'VAM', an initial BFS having the transportation cost Rs. 103 is given below :

Initial BFS

	1(3)				2(5)	3
3(5)			Δ (5)	1(7)		4 + Δ = 4
		2(3)				2
	2(6)	4(10)	2(5)			8
3	3	6	2 + Δ = 2	1	2	

Since the number of allocations (8) in the initial BFS is less than $m + n - 1$ (= 9), introduce negligible quantity Δ in the independent cell (2, 4).

							u_i
	+ (5) 2	1 • (3)	0 (7) 7	+ (3) 2	+ (8) 4	2 • (5)	−3
	3 • (5)	0 (6) 6	+ (12) 10	Δ • (5)	1 • (7)	+ (11) 8	0
	+ (2) −2	+ (1) −1	2 • (3)	+ (4) −2	+ (8) 0	+ (2) 1	−7
	+ (9) 5	2 • (6)	4 • (10)	2 • (5)	+ (10) 7	+ (9) 8	0
v_j	5	6	10	5	7	8	

Since all the net-evaluations are non-negative, the current solution is an optimum one. Hence the optimum solution is given by : $x_{12} = 1, x_{16} = 2, x_{21} = 3, x_{25} = 1, x_{33} = 2, x_{42} = 2, x_{43} = 4$ and $x_{44} = 2$.

The optimum transportation cost is

$$z = 1(13) + 2(5) + 3(5) + \Delta\,(5) + 1(7) + 2(3) + 2(6) + 4(10) + 2(5) = \text{Rs. } 103, \text{ as } \Delta \to 0.$$

Example 18. *A company has 4 warehouses and 6 stores; the cost of shipping one unit from warehouse i to store j is c_{ij}.*

If $C = (c_{ij}) = \begin{pmatrix} 7 & 10 & 7 & 4 & 7 & 8 \\ 5 & 1 & 5 & 5 & 3 & 3 \\ 4 & 3 & 7 & 9 & 1 & 9 \\ 4 & 6 & 9 & 0 & 0 & 8 \end{pmatrix}$, *and the requirements of the six stores are 4, 4, 6, 2, 4, 2 and quantities at the warehouse are 5, 6, 2, 9. Find the minimum cost solution.* **[IAS (Main) 95]**

Centres	A	B	C	D	E
Agra	55	30	40	50	40
Allahabad	35	30	100	45	60
Calcutta	40	60	95	35	30

Determine the optimal shipping cost.

[**Ans.** $x_{12} = 10, x_{13} = 20, x_{15} = 10, x_{21} = 20, x_{31} = 5, x_{34} = 30, x_{35} = 5$, min. cost = Rs. 3600].

4. A manufacturer wants to ship 8 loads of his product as shown in the table. The matrix gives the mileage from origin *O* to destination *D*. Shipping costs are Rs. 10 per load per mile. What shipping schedule should be used.

[**Hint.** Find the initial solution by using Vogel's method. In this solution, number of allocations is less than $m + n - 1$ (*i.e.*, 5). Hence resolve the degeneracy by introducing Δ to one of the empty cells [say, (2, 2)]. Then this initial solution will be optimal with minimum mileage 820 or cost Rs. 8200].

[**Ans.** $x_{11} = 1, x_{21} = 3, x_{32} = 2, x_{33} = 2$]

	D_1	D_2	D_3	Available
O_1	50	30	220	1
O_2	90	45	170	3
O_3	250	200	50	4
Required	4	2	2	

[IAS (Main) 91]

$$\sum_{j=1}^{n} x_{ij} = a_i , \qquad i = 1, \ldots , m$$

$$\sum_{i=1}^{m} x_{ij} \le b_j , \qquad j = 1, \ldots , n$$

$$x_{ij} \ge 0 , \qquad i = 1, \ldots , m ; j = 1, \ldots n .$$

Now, introducing the slack variable $x_{m+1,j}$ $(j = 1, \ldots , n)$ in the second constraint, we get

$$\sum_{i=1}^{m} x_{ij} + x_{m+1,j} = b_j , \qquad j = 1, \ldots ,n$$

or $$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} + x_{m+1,j} \right) = \sum_{j=1}^{n} b_j \quad \text{or} \quad \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right) + \sum_{j=1}^{n} x_{m+1,j} = \sum_{j=1}^{n} b_j$$

or $$\sum_{j=1}^{n} x_{m+1,j} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{n} a_i = \textit{shortage in availability } a_{m+1} \text{, say.}$$

Thus the modified general T.P. in this case becomes :

$$\text{Minimize} \quad z = \sum_{i=1}^{m+1} \sum_{j=1}^{n} x_{ij} c_{ij} ,$$

subject to the constraints :

$$\sum_{j=1}^{n} x_{ij} = a_i \quad , \quad i = 1, \ldots , m+1$$

$$\sum_{i=1}^{m} x_{ij} + x_{m+1,j} = b_j \quad , \quad j = 1, \ldots , n$$

$$x_{ij} \ge 0 \quad , \quad i = 1, \ldots , m+1; j = 1, \ldots, n .$$

where $c_{m+1,j} = 0$ for $j = 1, \ldots , n$ and $\sum_{i=1}^{m+1} a_i = \sum_{j=1}^{n} b_j$.

Working Rule : *Whenever $\Sigma a_i \le \Sigma b_j$, introduce a dummy source in the transportation table. The cost of transportation from this dummy source to any destination are all set equal to zero. The availability at this dummy source is assumed to be equal to the difference $(\Sigma b_j - \Sigma a_i)$.*

Thus, an unbalanced transportation problem can be modified to balanced problem by simply introducing a fictitious *sink* in the first case and a fictitious *source* in the second. The inflow from the source to a fictitious sink represents the surplus at the source. Similarly, the flow from the fictitious source to a sink represents the unfilled demand at that sink. For convenience, costs of transporting a unit item from fictitious sources or to fictitious sinks (as the case may be) are assumed to be zero. The resulting problem then becomes balanced one and can be solved by the same procedure as explained earlier. The method for dealing with such type of problems will be clear in *Example 19.*

Q. 1. What is unbalanced transportation problem ? How do you start in this case ?

2. Explain the technique used to solve the transportation problem with the following restrictions imposed separately in each problem :

(i) it is required to have all basic solutions non-degenerate,

(ii) it is required to have no allocation in the (i, j) th cell.

(iii) it is required to have some positive allocation in the (i, j) th cell.

(iv) it is found that the total units ready to ship be less than the total units required.

Example 19. *XYZ tobacco company purchases tobacco and stores in warehouses located in the following four cities :*

Warehouse location	Capacity (tonnes)
City A	90
City B	50
City C	80
City D	60